





BIBLIOTECA PROVINCIALE

Armadio

XV



Palchetto

Num.° d'ordine

27-9-8-21

115

B. L.

I

261



7,06.414 SBN.

MATHEMATICAL DISSERTATIONS,

FOR THE USE OF

STUDENTS IN THE MODERN ANALYSIS;

WITH IMPROVEMENTS

IN

THE PRACTICE OF STURM'S THEOREM,

IN THE

THEORY OF CURVATURE,

AND IN THE

SUMMATION OF INFINITE SERIES.



BY J. R. YOUNG,

PROFESSOR OF MATHEMATICS, BELFAST COLLEGE.



LONDON:

JOHN SOUTER, 131, FLEET STREET.

MDCCCXII.

PRINTED BY C. ADLARD, BARTHOLOMEW CLOSE.

PREFACE.

THE object which it is the intention of the present work to promote is briefly explained in the introductory article prefixed to the first Dissertation. This object, however, is but partially accomplished in the portion now submitted to the public ; the full completion of the original design is reserved for a *Second Part*, which will contain Dissertations on the doctrine of *Angular Sections*, on the *Theory of Imaginary Logarithms*, and on some *General Formulas of Analysis*.

The Part now published contains but four dissertations. The first of these is occupied with a short discussion of the theory of the coordinate signs ; mainly for the twofold purpose of removing the very strong objections advanced against this theory in the great work of CARNOT,* and of establishing, upon permanent and intelligible principles, the law which regulates the mutations of sign in the trigonometrical secant, and in the analytical expression for the radius of curvature.

This sign, which is a prefix purely algebraical, accountable for by the common theory of the algebraic signs, and independently of all convention as to their geom-

* Géométrie de Position.

of surfaces in general; systematically deduced from one very fertile and beautiful theorem—the theorem of the *Indicatrix*, an evanescent curve first imagined by DUPIN, and applied by him to very important purposes in the theory of curve surfaces.

Having deduced these general properties, I have in the next place attempted to give something like consistency and stability to the analytical theory of the *Lines of Curvature*, and of the remarkable points around which they circulate. This is a topic which seems not to have been discussed, as yet, with complete success. It is rather a delicate department of the higher geometry, and requires for its satisfactory development a very cautious use of the forms of analysis. Few persons since MONGE, the illustrious originator of these speculations, have expended any original effort upon the perfecting of this interesting theory. With the exception of DUPIN, who has added more to the labours of MONGE on this subject than any other person, and POISSON, who has recently examined the theory under a somewhat novel aspect, I know of no one who has contributed anything towards a satisfactory elucidation of the obscurities, or towards a removal of the discrepancies, with which this theory is still justly chargeable; yet this is one of the few of the more elevated speculations of geometers which find immediate application in the practical affairs of life.

In the profound work of DUPIN,* one main object of the

* *Développements de Géométrie.*

author was to correct the theory which MONGE had delivered on this subject; while the intention of POISSON's more recent *Mémoire** was to remove the defects which he conceived to attach to the expositions of both MONGE and DUPIN.

In the Dissertation which I have here given, I have ventured to discuss, with some freedom, the views and reasonings of each of these distinguished analysts, all of whom appear to me to have failed in their endeavours to place the theory of the curvature of surfaces beyond the reach of valid objection.

I think the chief difficulties and perplexities connected with the investigation of this theory will be found to lurk under the form $\frac{0}{0}$, the analytical symbol which has been a fruitful source of embarrassment and controversy in many other departments of analysis. The error commonly committed in the interpretation of this symbol appears to me to arise from not discriminating between the two distinct meanings, one or the other of which, according to circumstances, is really concealed under it.

In connexion with a certain restriction, this symbol is limited to a single value, or at most to a determinate number of multiple values. This restriction is, that the values of $\frac{0}{0}$ must obey the law implied in the general expression whence the form $\frac{0}{0}$ has arisen, upon introducing into that expression particular values for the arbitrary quantities.

* Journal de l'Ecole Polytechnique, 1823.

This is the condition which connects the particular value of $\frac{0}{0}$ with the entire series of values implied in the general expression, and which unites it with them by the common law which governs the whole. It is this particular value, to the exclusion of all others, which the *Differential Calculus* selects as the main instrument of its operations.

The criticisms, or perhaps more correctly the witticisms, of BERKELEY upon this subject in the *Analyst*, produced an unaccountable effect upon the mathematicians of his time, and shook the confidence of many in some of the most rigorous conclusions of analysis. BERKELEY, however, appears to have misunderstood, and thus to have misrepresented, the mathematical argument. He charged analysts with changing, at the close of the reasoning, the hypothesis upon which that reasoning commenced; whereas it was because of a rigid adherence to the hypotheses, even in the extreme and critical case of the problem, that the peculiarity of their result arose.

The *hypothesis*, in such a case as that before us, is, that each particular value in our general form must come under the law which binds the whole. The fixing a particular value at the close of the reasoning upon a quantity which had all along been considered arbitrary was certainly no *change* of hypothesis, since till then no hypothesis, as to value, had been introduced.

Results of the form $\frac{x^n - a^n}{x - a}$, as briefly noticed in the Introduction prefixed to the present volume, furnish the ex-

treme or critical case here adverted to upon putting $x = a$. When it is asked what this result *becomes* when $x = a$? the enquiry implies the twofold condition,

$$\left. \begin{array}{l} \frac{x^n - a^n}{x - a} \\ x = a \end{array} \right\}.$$

But the symbol $\frac{0}{0}$ may be capable of a much wider interpretation; it may be independent of extraneous restrictions, and be wholly indeterminate. This, however, it cannot be, unless the conditions that have led to it be indeterminate too; because the premises must implicitly contain the conclusion. It would be wrong, therefore, to offer an interpretation of such a result till we ascertain the effect of the hypothesis, to which it is due, upon our original conditions. The necessity for this examination has however been overlooked in the theory of the curvature of surfaces, as well as in some other analytical enquiries.

Another oversight, equally detrimental to the rigorous establishment of this theory, has also been committed by all preceding investigators; and which consists in neglecting the distinction between a certain *differential* equation, and the common *algebraic* equation into which it changes when the variables take constant values.

The substance of what I have ventured to offer, as corrections of the existing theory, was read before the Royal Society in December 1838. The publication of the essay has however been delayed, in the expectation that the long-promised reprint of the *Analyse Appliquée* of MONGE,

under the able superintendence of LIOUVILLE, would soon be presented to the public; and that it would probably appear with important modifications from the pen of its distinguished editor. But the French catalogues for 1840 merely repeat the announcement, so frequently made during the last five years, that the work is still "*sous presse.*"

The Dissertation which follows this is composed of two articles. The first contains an attempt to prove the incommensurability of the circle by help of only the common elementary formulas of analysis. But very few demonstrations of this interesting truth have been offered; and I believe not one that is entirely free from objection either on the ground of prolixity or inconclusiveness.

The second article is occupied with a new and simple method of finding the sums of certain infinite series whose values are expressible only in terms of the circular circumference; and which would thus seem manageable only by help of the more advanced algebraic theories, or by means of definite integrals. Their summation is shown in this article to lie within the powers of common elementary algebra. It may be proper to add that these investigations on Series were first published, a few years ago, in the *Philosophical Magazine*.

The fourth and last Dissertation in this first Part is upon a subject which, at the present moment, is exciting considerable interest among analysts, as well in this country as on the continent. I mean the THEOREM

OF STURM. I believe that I have already contributed somewhat to extend the knowledge of this important theorem among British analysts; and although it has been since disparaged and undervalued in certain quarters, I have always entertained the conviction that it must eventually supersede every other method at present known for effecting the complete analysis of a numerical equation. Sensible however of the objections to which it is fairly exposed, on account of the numerical labour frequently implied in it, I have been anxious to devise some method of so reducing the work as to render the operation practicable in moderately high equations.

I think I have, to a certain extent, succeeded in this very desirable object. At all events, by aid of the modifications recommended in this Dissertation, I have been able, with but a moderate amount of labour, to analyse complete equations of the fifth, sixth, and seventh degrees.

Although I have received no assistance from the work, yet I feel it incumbent upon me to state that I have seen a publication with the title "*Du Théorème de M. Sturm, et de ses Applications Numériques*, par M. E. MIDY; Paris, 1836;"—in which the general principle, upon which the operation for the common measure is conducted, is the same as that which I have adopted in the present Dissertation. This principle however is so trifling a modification of that by which the process is conducted in all the French books, that I question whether the author himself lays any claim to originality in the statement of it which he has given. My own improvements were matured, and

the details even completed, before the work of M. MIDY came to my hands.

These improvements, in so far as they are connected with the principle just referred to, entirely consist in the compact arrangement which has been given to the numerical calculation; to which arrangement the superior facility of the operation is entirely due. M. MIDY has not directed his attention to this circumstance; and indeed his object throughout has been rather to prove the impracticability of STURM's theorem beyond certain very narrow limits, than to promote its general adoption by the proposal of any method for facilitating its application. Such a proposal has however been more recently made by Professor DE MORGAN, in his article on "INVOLUTION" in the *Penny Cyclopædia*. It is pretty certain that the method of M. MIDY, that of Mr. DE MORGAN, and that proposed in the present work, are all referrible to one and the same general principle. The numerical labour however involved in this principle can be economized only by a judicious arrangement of the elements of the calculation.

Such an arrangement I have endeavoured to devise; and I have further reduced the labour by the adoption of an effectual method of restraining the large coefficients, which encumber and greatly complicate the closing steps of the operation, within moderate bounds.

In the course of the development of these views, several interesting points, connected with certain extreme or critical cases, naturally present themselves for examination.

But rather than digress into these matters, and thus interrupt the more direct order of proceeding, I have transferred them to the NOTES at the end of the volume ; in which I have endeavoured to anticipate the objections which might possibly be made to the efficiency of the general method in particular cases.

It is probable, from the facilities now introduced into the solution of numerical equations of the third and fourth degrees, and unfolded at pages 211 and 243, that this class of equations will hereafter be admitted even among the rudiments of algebraic science.

I cannot conclude this short account of the book without acknowledging that I have had some difficulty in fixing upon a suitable name for the miscellaneous collection of papers, of which it consists. Of the many titles that presented themselves to my thoughts, each was rejected in turn ; because the previous appropriation of it by some analyst of eminence could not fail to connect such title with associations unfavorable to a work of such slender claims to notice as the present. From this consideration the designations *Tracts*, *Disquisitions*, *Essays*, *Researches*, &c. were successively laid aside ; and the candid critical reader, who well considers the dilemma, will not I hope conclude, after the present explanation, that by calling this collection, " *Mathematical Dissertations*," I invite any comparison of it with the profound and important investigations of THOMAS SIMPSON.

J. R. YOUNG.

BELFANT COLLEGE; NOV. 25, 1840.

CONTENTS.

INTRODUCTION.

ART.	PAGE
1. Object of the present work	3
2. Importance and extent of the proposed enquiry	4
3. Instances of apparent discrepancies in the conclusions of algebra	5

DISSERTATION I.

On the Theory of the Coordinate Signs.

4. On the signs $+$ and $-$ considered in reference to their general interpretation	8
5. Introduction of these signs into geometry as symbols of direction	9
6. Statement of the objections of CARNOT to the geometry of DESCARTES	10
7. Inconclusiveness of CARNOT's first argument proved	13
8. Inconclusiveness of the second argument proved	14
9. Specific meaning of $+$ and $-$ in the theory of coordinates	16
10. On the meaning of these signs when prefixed to oblique lines	17
11. On the algebraic sign of the secant	18
12. How it happens that an oblique line, in a particular position, may have claim to one sign rather than to the other	19
13. On the error of supposing that the algebraic signs prefixed to geometrical quantities necessarily refer to position	ib.
ib. Objections to the common theory of the secant	20
14. On the sign of the radius of curvature	21
15. Proper method of expressing the radius of curvature in reference to sign	22
16. Methods of investigating this expression independently of the general theory of osculation	24

ART.	PAGE
17. Explanation of the principle on which the sign of the radius of curvature depends, and of that which gives rise to the double sign .	26
18. Easy method of obtaining the radius of curvature under a particular arrangement of the axes	27
19. Remarks on the conventional hypothesis of LACROIX in reference to the sign of the radius of curvature	28
ib. Note on the geometrical interpretation of imaginary quantities. .	29

DISSERTATION II.

On the Curvature of Surfaces.

20. On the principal contributors to the theory of curve surfaces	33
21. On the obscurity which still exists in one part of this theory	ib.
22. Views of DUPIN respecting the interpretation of $\frac{0}{0}$	34
23. Examination of these views	35
24. Fundamental mistake of MOND in reference to the lines of curvature	37
25. Remarks on the Mémoire of POISSON on the curvature of surfaces	38
26. Objections to the theory of POISSON	39
27. Investigation of the fundamental properties of surfaces	40
28. Form of the general equation for surfaces of the second order	42
29. On the evanescent curve into which a section of the surface merges when the cutting plane becomes tangential	43
30. This curve is the <i>Indicatrix</i> of DUPIN	ib.
31. Expression for the <i>ultimate ratio</i> of the square of a semidiameter of a section, to the distance between the cutting and tangential planes	44
32. Expression for the radius of curvature of a normal section through that semidiameter in terms of the aforesaid ratio	45
33. Theorem of EULER, expressing the connexion of any normal section with the principal sections	ib.
34. Sum of the curvatures of two rectangular normal sections shown to be constant	46
35. The radii of every pair of normal sections through <i>conjugate tangents</i> shown to be constant	ib.
36. Sections of mean curvature	47
37. Simple investigation of the theorem of MEUSNIER	ib.
38. Extension of the theorem of MEUSNIER	48
39. Expression for the radius of curvature in terms of the inclination of the trace of the normal plane to the axis of x , deduced from the <i>indicatrix</i>	ib.

ART.	PAGE
40. Conditions which determine whether the indicatrix be parabolic or not	49
41. Conditions which imply that the indicatrix is hyperbolic	50
42. Reason why in the latter case the curvature which is the least possible is not a minimum	51
43. Expression for the radius of curvature at an umbilical point	ib.
44. The indicatrix is only the curve of the second order which approximates to that in which the parallel sections vanish when the order of the surface is above the second	52
45. Consideration of the indicatrix in reference to surfaces of the second order	53
46. Enquiry whether osculating surfaces of the second order have other points in common besides the point of osculation	ib.
47. Remarkable property of the osculating paraboloid	55
48. General equation of the indicatrix	56
49. DUPIN's proof of the property in (47)	57
50. Inferences from the preceding property	58
51. Investigation of the differential equation of the lines of curvature	59
52. Equations of condition when $\frac{dy}{dx}$ has innumerable values	60
53. Impossibility of the two values of $\frac{dy}{dx}$ becoming identical	61
54. On the peculiar values of $\frac{dy}{dx}$ which fulfil more than the ordinary conditions	62
55. These values refer to directions of closer contact with the osculating sphere at an umbilicus	64
56. There is at least one direction in which the contact with the osculating sphere at an umbilicus is more than ordinarily close, unless the umbilicus is a vertex of revolution	65
57. At a vertex of revolution the contact is uniform all round the point	66
58. Method of ascertaining whether a surface has umbilical points	67
59. Method of determining the lines of curvature from the general differential equation	68
ib. Nature of the equation when the x and y in it receive particular values	69
60. Some of the conflicting doctrines about lines of curvature traced to a neglect of the distinction between the differential equation of the lines, themselves, and the algebraic equation involving the values of $\frac{dy}{dx}$ at particular points	ib.
61. Impossibility that the values of $\frac{dy}{dx}$, as given by the equation of the third degree, referring to an isolated umbilicus, can be all real	72

ART.	PAGE
62. Impossibility that any of these values can be imaginary when the umbilicus is not isolated, but situated on a line of spherical curvature	73
ib. Remarks on an erroneous inference of DUFIN	ib.
63. Probable cause of the preceding mistake	74
64. Observations on the cases in which the differential coefficients r, s, t , take the form $\frac{0}{0}$ at the point under consideration	75
65. How to ascertain whether the occurrence of this form actually implies the existence of points of a peculiar character	77
66. Investigation of the general equation of the lines of curvature on the ellipsoid	78
67. Particular equation of these lines when projected on the plane of the greatest and mean diameters	82
68. Remarks on the determination of the constant in this equation	b
69. Inference from this determination, namely, that through every point on the ellipsoid two lines of curvature pass	84
70. These two curves merge into one at certain points: there are four such points—the umbilici of the surface	85
ib. Erroneous deductions of MONGE, DUFIN, and POISSON on this subject	86
71. Remarks on the two values of the constant at each of the umbilici	ib.
72. On the true meaning of $\frac{dy}{dx} = \frac{0}{0}$ at an umbilicus. Errors of preceding analysts	88
73. On the actual projections of the lines of curvature: manner of constructing the elliptic projections	89
74. Construction of the hyperbolic projections	91
75. On the change which takes place at an umbilicus in the character of the surface	94
76. Equation of the unique line of curvature through an umbilicus deduced from the general equation	95
77. Lines of curvature on surfaces of revolution of the second order	96
78. Equation of the projections when the plane which receives them is that of the least and mean axes	97
79. On the actual construction of these projections	100
80. On the singular solution to the differential equation of the projections. Geometrical interpretation of this	101
81. On the lines of curvature of the hyperboloid	104
82. On those of the double-sheet hyperboloid	105
83. On the lines of curvature of the paraboloid	106
84. Nature of the projections when the paraboloid is elliptic	107
85. Character of the projections when the paraboloid is hyperbolic	108
NOTE on the sign of the radius of curvature of a normal section of a surface	110

DISSERTATION III.

On the Incommensurability of the Circle, and on the Summation of Infinite Series.

ART.	PAGE
86. On the different attempts to prove that the ratio of the circumference of a circle to its diameter is an interminable decimal . . .	116
87. Analytical proof of the Incommensurability of this ratio . . .	117
88. Formulas for the summation of certain classes of infinite series, with examples of their application . . .	120
89. Formulas for other classes of series, with examples . . .	128

DISSERTATION IV.

On the Theorem of Sturm, with Improvements.

90. Some account of the theorem	137
91. Comparative merits of the methods of LAGRANGE, BUDAN, FOURIER, and STURM	138
92. Investigation of a general form or type of the calculation . . .	142
93. Precepts to abridge the numerical work	146
94. Other means of facilitating the operation	147
ib. Examples of the method	148
95. Consideration of circumstances which sometimes diminish the number of functions	149
96. Analysis of an equation under these circumstances	155
97. Example of a method of shortening the numerical process, useful on certain occasions	157
98. Analysis of COLONEL TITUS's problem	159
99. Remarks on the preceding analysis, with additional examples of the method	161
100. Application of this method of analysis to literal equations of the fourth degree	165
101. Application of the method to literal equations of the fifth degree . .	167
102. Remarks on the general expressions which result from this analysis .	170
103. Application of the general expressions to a particular example of the fifth degree	171
104. On the general expression for X_2 in a literal equation of the sixth degree	172
105. Remarks on the increase of numerical labour attending the computation of the advanced functions of STURM	173

ART.	PAGE
106. On the means of reducing this labour; which is shown, however, to be less than that of any other method	174
107. Elementary character of the whole operation	175
108. Use of the transformations of <i>BUDAN</i> in certain cases: superiority of <i>STURM</i> 's method over that of <i>FOURIER</i> shown from the examples selected by <i>FOURIER</i> himself	178
109. Method proposed for abbreviating the work of computing the closing steps in <i>STURM</i> 's theorem	180
110. Example of the application of this abbreviated method	181
111. Remarks on the preceding operation	183
112. Analysis of the foregoing equation	185
113. Means of ascertaining whether roots, having several leading figures in common, are really equal or not	187
114. Application of the improved method to a complete equation of the sixth degree	189
115. Example of the analysis of an incomplete equation of the seventh degree	191
116. Application of the method to a complete equation of the seventh degree	194
117. Discussion of the case in which the derived functions do not descend in degree regularly by unity	196
118. Examples of the modification which the work undergoes in such a case	199
119. Example of the work when two consecutive functions differ in degree by three units	201
120. Example of the entire operation when the derived functions are fewer than the ordinary number in the analysis of a complete equation of the sixth degree	203
121. Brief summary of the facilities introduced into <i>STURM</i> 's method by the modifications proposed in this Dissertation	206
122. Observations on the rules of <i>FOURIER</i> and <i>BUDAN</i>	207
123. Proposal of a short method of performing the first step in <i>STURM</i> 's method; that is, of determining the function X_2	209
124. Investigation of this method	209
125. Application of this improved mode of working to the examples given in the <i>Theory of Equations</i>	210
126. Remarks on the extreme simplicity to which the analysis and solution of cubic equations is now reduced, and examples of the proposed improvement in equations of the higher degrees	212
127. Statement of the objects proposed to be accomplished in the <i>NOTES</i> subjoined to this Dissertation	216

ART.	PAGE
128. On the connexion between the coefficients in the functions of STURM, and those in LAGRANGE's equation of the squares of the differences	218
129. LAGRANGE's conditions for the general equation of the fourth degree	219
130. Simplification of one of STURM's conditions	221
131. Comparison of STURM's general conditions with those of LAGRANGE	223
132. Application of the method of proceeding, adopted in this Dissertation, to the theory of elimination	ib.
133. On equations containing equal roots, or roots nearly equal	225
134. Example of an incomplete equation of the sixth degree with equal roots	227
135. Example of a complete equation of the sixth degree with equal roots	229
136. Observations on the Infallibility of the Improved method, whether the roots be equal or not	232
137. On equations supposed to have roots differing from one another only at a remote figure	235
138. Proof that when equal roots exist, the function preceding the last always has its extreme terms divisors of the extreme terms of the proposed equation	236
139. On a certain principle of universal application in facilitating the analysis of an equation, and which has hitherto had but a limited scope	237
140. Partial application of this principle by STURM	238
141. Extension of it proved to be allowable	239
142. Remarks on the utility of the principle	240
143. Universality of it shown	241
144. Application of the principle to an equation of the seventh degree	ib.
145. Observations on the extreme simplicity to which the analysis of equations of the third and fourth degrees is now reduced	243
146. Supplementary Note on an improvement in a certain step of the investigation of the Binomial Theorem	245

The Author of the present work begs to announce that the
 ANALYSIS OF
 CUBIC AND BI-QUADRATIC EQUATIONS,
 according to the Improved Methods adverted to in Note II., and by
 means of which this part of the General Theory of Equations is reduced
 to a very remarkable degree of simplicity, will form the subject of a
distinct publication.

Also that a New and greatly Improved Edition of
 THE ELEMENTS OF
 PLANE AND SPHERICAL TRIGONOMETRY,
 &c. &c.,
 will be published shortly.

ERRATA.

Page 107, line 7, *for* hyperboloid *read* paraboloid.

119, 11, ... $A\pi^b$... $P\pi^b$.

DISSERTATION I.

ON THE THEORY OF THE COORDINATE SIGNS; OF THE SIGN
OF THE TRIGONOMETRICAL SECANT, AND OF THAT OF
THE RADIUS OF CURVATURE; WITH AN EXAMINATION
OF THE OBJECTIONS OF CARNOT.

MATHEMATICAL DISSERTATIONS.

INTRODUCTION.

(1.) THIS volume is intended for the use of those who may be prosecuting a course of study in the modern mathematical analysis ; and is undertaken in the hope that it may occasionally supply acceptable assistance and instruction supplementary to that furnished by our ordinary elementary works.

It is usually considered as not within the scope of such works to present more than a simple exposition of the elements of science. Controversial discussions and critical dissertations are professedly, and properly, excluded ; and indeed, one would think, absolutely forbidden in enquiries whose steps, being all conducted by the light of demonstration, must lead to conclusions necessarily true. Nevertheless, the fundamental principles of even the most firmly established analytical theories have been assailed and repudiated by mathematicians of the highest order. Discrepancies, still unreconciled, have been detected in our standard works on pure science ; and even some of the symbolical expressions most familiar to the algebraical student, are contemplated by modern analysts with difference of opinion as to their real interpretation.

(2.) It is quite clear that if these statements be true, it would be an interesting and, for science, an important undertaking, to institute such a searching enquiry into the sources of these discrepancies and conflicting doctrines, as would ultimately lead to the removal of all ambiguity, the reconciling of all apparent contradiction, and the establishing, beyond valid objection, of all those unsettled points which still divide the cultivators of modern science.

Such an extensive undertaking will not however be looked for in the present unpretending volume; which is designed merely to assist the young student in his efforts to acquire precision of ideas upon some of those more delicate points of analysis which present themselves even in an elementary course; but which are usually discussed without any cautionary hint to the reader as to the restrictions under which they must be received in order that they may conform to the prescribed laws of algebra.

(3.) If one could be quite certain that these governing conditions would be always present to the mind when employing the forms of analysis, any such caution would be superfluous; but, as the preceding statements imply, the necessary controlling influences are sometimes overlooked; and thus contradictory theories and disputed doctrines have arisen, which have tended to bring discredit on the conclusions of algebraic science; for there can be but little doubt that every discrepancy in the results of analysis must arise from some oversight of this kind, however difficult it may be to detect the precise step in the investigation where the inadvertence has occurred. In the extreme case of a problem, where the hypothesis is considered as having reached its ultimate state; and, on the other hand, in those numerous extensions of analytic formulas by which they are made to comprehend cases even beyond the bounds of the original hypothesis, especial caution is necessary in order to avoid mistakes of this kind; because

in these critical or novel circumstances, our formula may be materially affected by some of those general laws of analysis which, in the ordinary interpretation of the expression, are not brought into operation; and which are therefore in danger of being disregarded when our modifications really render such expression wholly subject to their influence. To illustrate our meaning by a simple instance, we may remark that the signs of operation + and - are wholly inefficient when prefixed to zero or 0; so that when an additive or a subtractive quantity has, in the ultimate state of the hypothesis, been made to disappear, it is, in ordinary cases, of no moment whether we preserve any trace of this quantity by the actual insertion of the zero in its place or not; in fact, such insertion would usually and justly be considered as a useless incumbrance, and the zero is accordingly suppressed; yet there are cases in which this suppression would lead to what, for the moment, would seem a contravention of the established principles of algebra. Thus let the following quadratic equation be proposed for solution, namely,

$$x^2 - 4x = -4;$$

then completing the square, we have

$$x^2 - 4x + 4 = 0,$$

and extracting the root

$$x - 2 = 0$$

$$\therefore x = 2.$$

Now we know it to be a principle, perfectly general and incontrovertible, that every quadratic equation has *two roots*; whereas we are here presented with only one, and no trace is exhibited of a second. The apparent discrepancy will no doubt be easily accounted for by the reader, who will see that if the zero which is here suppressed be inserted, both the roots will present themselves in the

result. Restoring then this zero, we shall have, after extracting the root,

$$x - 2 = + 0 \text{ or } - 0,$$

hence the two roots are

$$x = 2, x = 2,$$

that is, the roots are equal, each being 2.

Again, if we have a fraction, such as $\frac{x^3 - a^3}{x - a}$, and we require to find the value of it in a particular case, as for instance when $x = b$, b being different in value from a , we may at once substitute b for x , perform the division, and the sought value will be exhibited in the quotient. It is easy to see that this value is to a certain extent independent of the form of the general fraction; for it might, as far as this particular case of it is concerned, be changed into $\frac{bx^3 - a^3}{x - a}$, or $\frac{b^3x - a^3}{x - a}$. But take the *extreme* case, that

of $x = a$, then the corresponding value of the fraction will be entirely governed by its form; and will no longer be the same under the changes $\frac{ax^3 - a^3}{x - a}$ and $\frac{a^3x - a^3}{x - a}$. Here then is an instance where the law, by which a general algebraic expression controls the individual cases comprehended in it, is rigorously enforced only in the extreme one of those cases, in which a neglect of its influence would lead to error, although no error would arise from such neglect in any other of the series of cases which this one terminates. The three fractions

$$\frac{x^3 - a^3}{x - a}, \frac{ax^3 - a^3}{x - a}, \frac{a^3x - a^3}{x - a} \dots\dots [1]$$

furnish the three different quotients

$$x^3 + ax + a^2, ax + a^2, a^3 \dots\dots [2]$$

and, for $x = a$, the three different values

$$3a^2, 2a^2, a^2;$$

the first of which is the only value belonging to the series of values comprehended in the expression $\frac{x^3 - a^3}{x - a}$, under the various hypotheses as to the value of x . This is plain, because whatever be the law implied in either of the expressions [1], the same must be implied in the identical expression [2]; this latter being the real thing that [1] represents; and only the *first* of [2] furnishes the proper value of the proposed fraction for $x = b$.

Having made these introductory remarks, we shall now proceed to examine some of those more interesting parts of analysis respecting which the student may be in danger of entertaining mistaken views; either from the subtlety of the reasonings which they involve, or from the force of the more formidable objections which may have been brought against the principles upon which they rest. Where these principles seem to us to be *really* defective, or the arguments founded upon them illogical, we shall endeavour to make the blemishes apparent, and shall supply, as well as we can, the necessary corrections.

DISSERTATION I.

ON THE THEORY OF THE COORDINATE SIGNS.

(4.) IN the modern application of algebra to geometry a very important and novel use is made of the signs $+$ and $-$, the marks which in calculation usually denote the common operations of addition and subtraction. We say usually denote, because, although such is their primary office, and the purpose for which they are conventionally introduced into algebraical notation, in the character of *signs of operation*; yet, as every student of algebra knows, these same signs, while they perfectly perform the office originally allotted to them, are moreover efficient beyond the limits which this office seems to prescribe to them; and admit of a wider interpretation than appears to be implied in their original definition.

This apparent extension of the meaning given to these symbols at the outset is so naturally suggested to us after we have advanced a few steps, and is seen to be so necessary to the perfection of algebraic language, that it is usually at once tacitly made, so soon as the need for it is felt, without formally returning to our definitions for the purpose of making the modifications apparently rendered necessary. Thus, the meaning of the isolated negative quantity, whenever it presents itself in the result of a problem, at once suggests itself; and is dependent upon no new convention. A *subtraction*, in the common sense of the word, is certainly not denoted by the *minus* in such a case; yet it must bear to the *plus* a relation perfectly

analogous to that which subtraction bears to addition, otherwise our meaning would be something more than a simple extension of that already laid down; it would be an incongruous addition to it. The isolated minus quantity therefore, standing thus in direct opposition to the isolated plus quantity, has its meaning immediately inferrible from that of this latter, which is always at once fixed by the problem itself. If, for instance, the problem be to find the amount of gain arising out of any pecuniary transaction, this gain will be represented algebraically by an isolated quantity assumed to be *plus*; but should the result not justify this, but turn out to be *minus*, the proper interpretation, in strict conformity to the principle laid down in the definitions, of the direct opposition of $+$ and $-$, is obvious; the result of the transaction in question is not *gain* but *loss*. In like manner if the enquiry respect the distance of an object from any fixed boundary, and we start by assuming the object to be upon one side of the boundary and at a distance from it, represented as usual by a quantity assumed to be *plus*, which however appears in the result with *minus*, we immediately infer, and in strict conformity to the principles laid down, that, although the *distance* of the object from the boundary was properly represented, its place is on the side of it *opposite* to that supposed.

(5.) It thus appears that the signs $+$ and $-$, though attached to isolated quantities, and consequently not implying any addition or subtraction, have nevertheless a meaning which is perfectly intelligible; and which, though quite distinct from the signification primarily given to them, is in strict harmony with that signification, and is in fact only an extension of it.

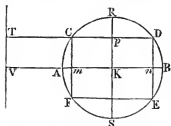
It was the observation of this circumstance, coupled with considerations still more refined, that doubtless led DES CARTES to his beautiful system of analytical geo-

metry; in which the algebraic letters x, y , being employed to represent the absolute distances, *abscissas* and *ordinates*, of the points of a curve from two fixed boundaries or *axes* crossing one another, the algebraic signs $+$ and $-$, prefixed to these letters, make known upon which side of each boundary these several points are situated; and thus can be expressed in the language of algebra, not only *distance* but *position*.

In *which* direction from any boundary or axis the distance of a point shall be marked $+$, is obviously a matter of arbitrary choice; but this choice once made, then the mark $-$, conformably to what has been shown above, necessarily implies distance in the *opposite* direction. The only thing therefore that is conventional in this appropriation of the algebraic signs, is *which* of the two directions shall be called plus, the opposite direction being minus as matter of course. We presume the student to be already sufficiently familiar with the principles of analytical geometry to be aware of what the ordinary convention is as to the disposal of the plus sign; that is, in what directions the positive abscissas and the positive ordinates are usually taken: he will do well also to remember that, in this application of the signs, the terms *positive* and *negative* are merely names given to these *symbols of direction*, and that they imply no distinction among the lines themselves save that of direction.

(6.) CARNOT, a mathematician of great eminence, has strenuously opposed, as inadmissible and absurd, the fundamental principles, and consequently the entire system of the Cartesian geometry. We are not aware that his reasonings on this matter have ever been proved to be nugatory. But we are confident that, if the remarkable work in which his objections are developed at such length, were generally accessible to or consulted by students entering upon the study of analysis, but few would care to waste

their time on the alleged absurdities of analytical geometry. To prove the existence of these absurdities CARNOT reasons as follows:*



Let there be any curve, a circle for example, ACRDBESF, in the plane of which let two axes AB, RS, be traced perpendicular to one another, and cutting at the centre K, taken for the origin of the coordinates. From any point C in this curve draw Cp , and prolong it to D. Now, from the foregoing principles, pD being regarded as positive, Cp will be negative, so that we ought to have $Cp = -pD$; whence we get $Cp + pD$, or $CD = 0$, which is absurd. In like manner, if from the same point C, we draw Cm perpendicular to AB, and prolong it to F, we shall have $mF = -Cm$; consequently, $Cm + mF$, or $CF = 0$, which is equally absurd. Moreover, since we have got $CD = 0$, $CF = 0$, we shall have also $CD \cdot CF = 0$; that is to say, the area of the rectangle CDEF = 0; from which it would follow, for example, that the square inscribed in a circle would be 0. Such are the errors which unavoidably arise from the foregoing notion. It is then false that Cp is a negative quantity; it is easy to prove, indeed, that it is necessarily positive, for if from any point V of the straight line AB prolonged beyond A, we draw a straight line VT parallel to KR, it is evident, and admitted even by those who adopt the preceding notion, that the three straight

* *Géométrie de Position—Dissertation Préliminaire*, p. x.

lines TC, Tp , TD, all on the same side of the new axis VT, are positive: now we have $Cp = Tp - TC$; therefore, since $Tp > TC$, $Tp - TC$, or Cp , is a positive quantity.

It may, perhaps, be said that Cp is positive in reference to the axis TV, and negative with respect to the axis RS. But this would be a new enigma to solve more difficult even than the first, and that too in a science whose principal character is evidence.

To demonstrate in a manner still more palpable how by this principle of negative quantities taken in a direction contrary to that of the positive quantities, we are unavoidably led into error, I shall give a new form to the preceding argument as follows:

Let the circle ARBS be described, let K be its centre, and draw a diameter AB; we shall have

$$AB = AK + KB \dots [A].$$

Now through the centre K, I draw the axis RS, perpendicular to AB; then through any point C of the circumference, I draw the chord CD parallel to AB, and which I suppose to cut the axis RS in the point p . I take R for the origin of the abscissas: I call the abscissa Rp , x ; the ordinate pD , y ; and the radius a . I shall then have

$$y^2 = 2ax - x^2, \text{ and therefore } y = \pm \sqrt{2ax - x^2};$$

which teaches me, from the admitted theory of negative quantities taken in a contrary sense to the positive quantities, that y has two values, equal and directly opposite: one pD represented by the positive root, the other Cp represented by the negative root; that is to say, we have

$$pD = + \sqrt{2ax - x^2} \text{ and } Cp = - \sqrt{2ax - x^2}$$

equations which ought to have place whatever be the value of Rp or x . Let us suppose then that $Rp = RK$, or $x = a$; pD will then become KB, and Cp , AK; there-

fore the preceding equations will be $KB = +a$, and $AK = -a$. Substituting these values of AK , KB in the equation [A] found above, we shall have $AB = 0$; a result which is absurd, though rigorously deduced, or rather *because* rigorously deduced, from the theory of negative quantities being contrary in direction to positive quantities. This theory then is completely false. It is possible perhaps to oppose metaphysical subtleties to the above reasoning; but I do not believe that it can be answered in a manner clear and satisfactory to a geometrical mind.

(7.) Such are the reasonings and conclusions of CARNOT. The student will do well to examine them with that attention which the arguments of so distinguished a man must always deserve; and in proportion as this examination unsettles his confidence in those theories which he had previously thought to have been firmly established in his own mind, will be the necessity of a return to, and a vigilant scrutiny of, the principles whence they have been deduced, and which are here so unsparingly condemned.

But in all these arguments CARNOT has committed the singular oversight of making assumptions in his premises which are themselves distinct contradictions of the very theory he impugns; and, as his reasonings from these premises are perfectly logical, the conclusions must be absurd as matter of course, provided the theory contradicted in the premises be really accurate.

For, referring to his diagram, and bearing in mind the theory of the directive signs $+$ and $-$, the two lines pD , pC , in reference to the boundary or axis RS , are respectively represented both in absolute length and in direction by $+pD$ and $-pC$. If we wish merely to consider the absolute lengths of these lines with a view to ascertain that of their sum or difference, then, as direction is no element of the calculation, the signs of direction may, and indeed ought, to be suppressed. But if we choose to

preserve them we recognize their theory; and must of course conform to it from the outset. Hence, although the length of pD is equal to that of pC , yet it is not true that $+pD = -pC$; the sign of equality palpably contradicts the statement implied in the signs $+$ and $-$, that the directions are *not* the same, but opposite. CARNOT, in the equation $Cp = -pD$, with which he sets out, affects to express, agreeably to the theory, that Cp is equal and opposite to pD ; and thus the sign $-$ is prefixed to pD to overrule this contrariety of direction, and so render the equality complete. Now this would be quite unobjectionable if the lines Cp , pD of the figure, were really represented, in the foregoing algebraical expression of them, with the proper directive signs; but these signs, which involve the whole matter in dispute, are unaccountably omitted; and the equation therefore is not that which the theory condemned would furnish. In accordance with this theory the proper representation of Cp , as well in direction as in magnitude, is $-Cp$; and that of pD , $+pD$; and either of these is strictly equal to the other, when the direction of one of them is reversed, or that reversion assumed by changing the sign of either. The correct equation is therefore $-Cp = +pD$; from which, of course, no absurdity can arise. A similar oversight is committed in the statement that $mF = -Cm$, where the right direction is given to Cm and the wrong one to mF .

(8.) Let us now briefly examine the other argument, in reference to the equation [A], namely,

$$AB = AK + KB \dots [A],$$

the second member of which does correctly express the absolute distance between the points A and B: the directive sign of AK is however not actually exhibited; nor would this be matter of censure, but rather of approval, since, as we have already remarked, when the object is to exhibit

merely the absolute sum or difference of lines, the directive signs are superfluous. As, however, the equation is brought forward in the subsequent reasoning as if the proper directive signs were actually inserted, a supposition is made with respect to that equation which is contrary to fact: the sign of AK is not there. Introducing this sign, the proper mode of expressing the interval AB will be

$$AB = -(-AK) + KB \dots [B]$$

and, as the absolute length of AK, or KB, is assumed equal to a , we have $KB = a$, and $-AK = -a$; and therefore $AB = 2a$, as it ought to be.

Now it is worthy of observation, that though the double minus in [B] really expresses the same thing as the plus in [A], yet the form [B], of expressing the interval AB, in conformity to the theory we are here discussing, is immeasurably superior to the form [A], adopted by CARNOT, since this latter is not general, but requires that the point A be never situated nearer to B than K. If A were to move up to K and pass through it, [A] would no longer represent the interval AB: the diagram would show us that that expression must be modified before AB could be accurately represented by it. We should thus have to depend upon our diagram as a check to prevent errors we might otherwise commit if guided solely by our algebraical representation: that representation then is imperfect; and it is the paramount advantage of the coordinate method that from this imperfection it is entirely free; it completely pictures the diagram in the algebraical symbols, and all its deductions may be interpreted, with unerring certainty, independently of any aid from the figure to which those deductions apply. In the method of coordinates the interval (AB) between the extremities of any two abscissas x' , x'' is *always* expressed by $AB = x' - x''$; where x' , x'' stand for the lines both in magnitude and direction: in the particular case, [B], x'' has the particular value $-AK$.

It is thus easy to see, without any more minute exa-

mination of CARNOT's objections, that they are founded in an erroneous view of the doctrine opposed.

(9.) It may be well to remark here, as it is of consequence that the student should always bear the fact in remembrance, that in thus introducing $+$ and $-$ as signs of direction into geometry, we recognize the coordinate directions only; and affirm nothing about the signs of lines oblique to the coordinates, nor even of lines that may be detached portions of the coordinates themselves. The directive signs employed in analytical geometry originate in the assumption that lines measured from the origin along either of the fixed axes in a specified direction, are $+$; and in the inference from this assumption, and from the mutual relation of $+$ and $-$, that lines from the same origin along the same axes, but in opposite directions, are $-$. The directive signs, be it observed, have no other appropriation in analytical geometry but this; and if any line, not an abscissa nor an ordinate, be affirmed to be plus or minus, the assertion cannot be justified as simply a consequence of the principles already laid down, but only in virtue of some new assumption or additional principle. The question so frequently agitated about the directive signs of lines oblique to the coordinates is one of great absurdity. No line has any such sign till we arbitrarily endow it with one, or, which amounts to the same thing, till we give one to the line directly opposite to it. In enquiring, therefore, about the directive sign of a line, the first question to be put is,—“Is that line an ordinate or an abscissa?” If it be one or the other the enquiry is a proper one, admitting of a definite answer, for a determinate directive sign such line unquestionably must have; but if the line be neither an ordinate nor an abscissa, the enquiry is not only useless but meaningless, no directive sign having been appropriated to it.

Whenever, therefore, we have to determine analytically

the length of a line oblique to the coordinates, the sign or signs with which the resulting absolute value may be accompanied, arise out of the very general processes of algebra; and in the particular case in question are, as far as direction is concerned, perfectly meaningless appendages, in the absence, be it remembered, of all special assumption or purely arbitrary interpretation. It may, however, be noticed that when such a result appears with the single sign — it will be found, upon an examination of the reasoning, that an assumption has at some part of it been made, and perhaps made inadvertently, of the correlative +; but for such an examination there is no real occasion, as the sign has no influence upon the absolute length, and this is the only thing that is strictly determinate.

(10.) Nevertheless, as respects those oblique lines which, while they preserve their general relation to certain coordinates, are susceptible of continuous changes of length and position, and so pass through zero or infinity, we are not always at liberty to prefix to them in any position either of the signs + or — indifferently, nor to deny their strict claim to one of these signs rather than to the other; not, however, because these signs refer to the *direction* of the varying line in any proposed position; they are not directive signs of the line at all: but because in the passage of the line through 0 or ∞ , its algebraic sign *may* have changed. It may be sometimes easy to see whether or not such changes really take place; if they do take place, then it is plain whatever sign the line took before, it must take the opposite sign after the passage; so that knowing the algebraic sign that is prefixed to it in one position, and knowing where these changes happen, we can, in following the progress of the varying line up to any other position, always pronounce upon its algebraic sign.

(11.) These are the considerations which ought to guide

us in determining the algebraic sign of the secant in analytical trigonometry. This is not a *directive* sign as is usually implied; its changes arise from mutations of absolute value, and not from variations in position. The pre-established conventions however, provided these are in strict conformity to the general laws of algebra, can alone regulate the sign (otherwise arbitrary) which the *initial* value of the secant ought to take. After this initial sign is thus fixed, the changes take place, irrespective of all convention, in obedience to the law that the algebraic sign prefixed to any varying magnitude cannot change unless the magnitude pass through either 0 or ∞ .*

In proceeding through the first quadrant the secant, in virtue of this law, can undergo no change of sign; and that it may at the same time conform to the previous convention, when it coincides with the coordinate axes, it is necessary that this unchanged sign be *plus*. Arrived at the extremity of the quadrant it is $+\infty$, the opposite secant being $-\infty$. It is this latter that now moves, intersecting the negative tangent, diminishing in length up to the initial position, and thence increasing to infinity. Throughout all this range the law forbids a change of sign; the secant therefore continues *minns*. The opposite secant, taking of course the opposite sign, now revolves, and so on continually; the coordinate conventions as to *which* directions on the axes shall be plus, being regarded only so long as the fixed laws of algebra, paramount to all conventions, permit. Similar reasoning applies to the chord. Its initial position, corresponding

* The converse of this however is not necessarily true; for although a change of sign can arise from no other cause than a passage through 0 or ∞ , yet a varying quantity may pass through either of these states without undergoing any change of sign; and on this account, as remarked by CAENOTT, (*Trigonométrie*, seconde édition, p. 12,) the principle stated in the text is not alone sufficient to establish the mutations of sign in the varying trigonometrical lines. The passage through 0 or ∞ must be accompanied with opposition of direction to necessitate a change of sign.

to the arc $+0$, has the direction of the plus tangent; the revolution of this tangent will give all its other positions round the circle; and as it continues finite till the revolution is completed the values are all *plus*. At the end of the revolution, however, the value is again 0; but since it is still plus, and as the series recommences in the direction opposite to that where this terminates, it recommences with a *minus*. The value therefore in passing through 0 changes its sign from $+$ to $-$ and continues $-$ till it again passes through zero, and so on.

(12.) It is easy to see how it happens that, in the case of such oblique lines as we have been here considering, the value in any particular position has claim to one sign rather than to the other. Every such line is but one of a continuous series of lines, united together by a common law, and commencing at a common fixed origin, and each is estimated in reference to this union and to the common origin. The original sign is fixed by the coordinate convention, the changes by the common law of the signs.

(13.) It is a mistake to suppose, as CARNOT uniformly does, that the particular appropriation of the signs $+$ and $-$, to mark the coordinate directions, necessitates the interpretation of these signs as marks of direction in general. On the contrary, they always preserve their purely algebraic character in every analytical enquiry, implying direction only when prefixed to coordinates in conformity to the original convention, and which convention is quite consistent with their common algebraic import.

Instead, therefore, of seeking for an explanation of the minus sign prefixed to any isolated line in the particular direction that line may take, unless indeed the line be an ordinate or abscissa, we ought to regard such sign as having no reference to *position*, but simply as a consequence of some previous supposition of plus as to *value*, in the course

of the investigation whence our minus result has arisen. Cases of this kind have nothing to do with the coordinate theory, which they neither confirm nor oppose; their consideration belongs to the common theory of the algebraic signs in their ordinary character of signs of operation.

It may be proper to observe, that although, in speaking of the signs of the trigonometrical lines, we have considered these lines only in reference to the positive arcs, yet the very same arguments hold when the arcs are negative, proceeding from the origin in the opposite direction. At the origin itself the secant belongs indifferently to the first of the positive, or to the first of the negative series of arcs; and, as in passing through this state from either side, no change of sign can take place, we infer that, whether the arc be positive or negative, the sign of the secant of it is the same.

The common expression for the secant is $\frac{R^2}{\cos}$, which truly represents it both in value and algebraic sign; for the secant not only passes through ∞ when the cosine passes through 0, but it so *happens* that it also changes sign after this passage. The theory which connects the sign with the *direction* of the secant must, to be consistent, give the same sign to the radius whose prolongation that secant is. The R^2 is not the square of this radius, but the combination of the initial radius with it. Hence, when the secant is negative, R^2 ought to be negative too; and this would lead to absurdity.

Although at the origin the secant is the same, whether that origin be viewed in connexion with the positive or with the negative arcs, yet it is not so with the cosecant, which is $+\infty$ in the former case, and $-\infty$ in the latter. A like ambiguity is observable in the cotangent.

But the ambiguity here noticed, although more palpably felt than in the other trigonometrical lines, is nevertheless not peculiar to the cosecant and cotangent. These

others are all subject to it at their limits, although all situated in the co-ordinate directions. For take any finite ordinate, say a *plus* ordinate, and let it continuously shorten, till its extremity reaches the axis of abscissas; its representation is then strictly $+0$, while the *opposite* ordinate, diminishing in the same manner, and arriving at the same limit, would be -0 . To be sure, as the value here actually *vanishes*, attention to this difference of sign is commonly of no moment; yet the analytical fact of the existence of such difference ought not to be overlooked; and indeed, in the doctrine of vanishing fractions, the neglect of it might unquestionably lead to error.



ON THE SIGN OF THE RADIUS OF CURVATURE.

(14.) In the general theory of curves, founded upon that of coordinates, the differential expression for the radius of curvature is usually written thus :

$$R = \pm \frac{\left\{ \frac{dy^2}{dx^2} + 1 \right\}^{\frac{1}{2}}}{\frac{d^2y}{dx^2}} \dots [A]$$

the double sign being introduced because of the extraction of the square root, and the proper sign to be taken, in any individual case, is usually supposed to be such as to mark the direction of curvature. But the radius of curvature, being in general a line oblique to the coordinates, has, in strictness, no directive sign; and therefore to

designate it as positive or negative, in reference to *direction*, is improper. The only thing that is determinate is the *absolute length* of the radius. That which determines direction of curvature is the algebraic sign of $\frac{d^2y}{dx^2}$; for if a linear tangent be drawn through the point of osculation (x, y) , an ordinate to it, corresponding to the abscissa $x + h$, is

$$y + \frac{dy}{dx} h \dots [1]$$

and an ordinate to the curve, corresponding to the same abscissa, is

$$y + \frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{2} + \&c. \dots [2]$$

so that taking a sufficiently small value of h , [2] will obviously exceed or fall short of [1] according as $\frac{d^2y}{dx^2}$ is positive or negative; in other words, the curve will be convex to the axis of x at the point of contact, if $\frac{d^2y}{dx^2}$ be plus, and concave if it be minus.*

(15.) Because in the common applications, it is the part of the curve concave to the axis of x that is usually submitted to the above formula, it is sometimes recommended to preserve in that formula only one of the two ambiguous signs, the sign *minus*, in order that the radius belonging to

* All this however is on the supposition that the point under consideration is itself situated *above* the axis of x , or has a positive ordinate. The *general* inference from [1] and [2] is that the curve will be convex to the axis of x if y and $\frac{d^2y}{dx^2}$ have the same signs, and concave if they have different signs.

a concavity may come out plus.* But these subsequent adjustments of a general formula can never be necessary unless there be some defect in the analysis, or a mistake in the result itself; and accordingly, by paying strict attention to the algebraic signs throughout the general investigation whence R is derived, it will be invariably found, whatever method of reasoning be employed, that $\frac{d^2y}{dx^2}$ enters into the resulting expression for R preceded by the sign *minus*; so that, altogether disregarding the double sign $\pm$, which is a meaningless appendage to the above form, the proper value of R is

$$R = \frac{\left\{ \frac{dy^2}{dx^2} + 1 \right\}^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}} \cdot \cdot \cdot \cdot [B]$$

If this formula be applied to any particular point in the *convexity* of a curve we shall have a *negative* result, but this is merely one of those common algebraical cases already adverted to (13), and is to be explained in the same way; namely, we are to conclude that in the particular example before us, a negative quantity, or a negative remainder, has in some part of the investigation been tacitly considered as positive. Upon examination we accordingly find that $y - \beta$, which occurs in every step of the reasoning, and which is positive for the concavity,

* "Le double signe est relatif à la position de R ; si la courbe tourne sa concavité vers l'axe des x , alors $\frac{d^2y}{dx^2}$ sera négatif, et pour qu'alors R se détermine positivement, nous le prendrons avec le signe négatif, et nous écrirons

$$R = - \frac{\left(1 + \frac{dy^2}{dx^2} \right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}."$$

(Boucharlat, *Calcul. Diff.* p. 112.)

changes to negative for the convexity; so that if the change take place for two points in the same curve, R , in proceeding from one to the other, must pass through ∞ .

We may with propriety affirm, therefore, that when, in any individual example, R comes out plus, not that this is in consequence of R taking a particular direction, but that the curve is concave to the axis of x at the point; and that when it comes out minus the curve is convex.

(16.) The following methods of investigating the differential expression for the radius of curvature, independently of the general theory of osculation, may perhaps be acceptable to the student.

The trigonometrical tangent of the inclination of a curve at (x, y) to the axis of x is

$$\tan a = \frac{dy}{dx}$$

and, consequently, the tangent at the point whose abscissa is $x + h$ is, by Taylor's theorem,

$$\tan a' = \frac{dy}{dx} + \frac{d^2y}{dx^2}h + \&c.$$

Hence, by the known trigonometrical form for the tangent of the difference of two angles we have, calling $a' - a$, Δa

$$\tan \Delta a = \frac{\frac{d^2y}{dx^2}h + \&c.}{1 + \frac{dy^2}{dx^2} + \frac{dy}{dx} \frac{d^2y}{dx^2}h + \&c.}$$

The angle Δa is that between the two linear tangents at the points whose abscissas are x and $x + h$; or which is the same thing, the angle between the normals at those points. Call this latter angle $\Delta \theta$; then, observing that

$\tan \alpha$ and $\tan \theta$ have contrary signs, and increase or diminish together, we must have

$$\frac{\tan \Delta \theta}{h} = - \frac{\frac{d^2 y}{dx^2} + \&c.}{1 + \frac{dy^2}{dx^2} + \frac{dy}{dx} \frac{d^2 y}{dx^2} h + \&c.}$$

Consequently, in the limit,

$$\frac{d\theta}{dx} = \frac{-\frac{d^2 y}{dx^2}}{1 + \frac{dy^2}{dx^2}} = - \frac{\frac{d^2 y}{dx^2}}{\frac{ds^2}{dx^2}}$$

Now $\frac{ds}{d\theta}$ expresses the radius of curvature,* and this is the

same as $\frac{ds}{dx} \div \frac{d\theta}{dx}$; consequently,

$$R = \frac{\left(\frac{ds}{dx}\right)^2}{-\frac{d^2 y}{dx^2}} = \frac{\left\{\frac{dy^2}{dx^2} + 1\right\}^{\frac{1}{2}}}{-\frac{d^2 y}{dx^2}}$$

Otherwise,

$$\tan \alpha = \frac{dy}{dx}; \text{ and } R = \frac{ds}{d\theta}.$$

Differentiate the first of these, and we have, observing as

above that $\frac{d\alpha}{dx} = -\frac{d\theta}{dx}$,

$$\sec^2 \alpha \frac{d\alpha}{dx} = -\sec^2 \alpha \frac{d\theta}{dx} = \frac{d^2 y}{dx^2}$$

* For Δs being the arc of the curve subtending the angle $\Delta \theta$ between the normals at its extremities, it is plain that $\frac{\Delta s}{\Delta \theta}$ continually approaches to, and at length arrives at R ; as this ratio proceeds towards, and finally arrives at its ultimate state, $\frac{ds}{d\theta}$.

$$\therefore \frac{d\theta}{dx} = - \frac{\frac{d^2y}{dx^2}}{1 + \tan^2 \alpha} = - \frac{\frac{d^2y}{dx^2}}{1 + \frac{dy^2}{dx^2}}$$

$$\therefore R = \frac{ds}{dx} \div \frac{d\theta}{dx} = \frac{\left(\frac{ds}{dx}\right)^2}{-\frac{d^2y}{dx^2}}$$

as before.

(17.) To whatever particular example this general formula for R be applied, if the result come out plus, we may infer that, whatever tacit supposition as to this sign has been apparently introduced into the general investigation of R, the particular case in question confirms that supposition; but if the result come out minus, then we are to conclude that the supposed plus is in our particular example minus. In the general investigation given in the *Differential Calculus*, page 164, the quantity subject to this change of sign, and yet apparently assumed plus, is $y - \beta$. In the first investigation above it is $\alpha' - \alpha$; and it is virtually this in the second process, the assumption being that $\frac{da}{dx}$ is plus. Let either of these suppositions be reversed, and R will change its sign accordingly. We may of course leave the hypothesis distinctly open throughout the general investigation: this will merely require that we write $\pm (y - \beta)$ for $y - \beta$; or $\pm \Delta \alpha$ for $\Delta \alpha$; or $\pm \frac{da}{dx}$ for $\frac{da}{dx}$; we shall then arrive at the double formula

$$R = \frac{\left(\frac{ds}{dx}\right)^2}{\mp \frac{d^2y}{dx^2}} \dots [C]$$

the upper or lower sign having place according as the upper or lower sign has place in the hypothesis, as to the quantities selected above; or, which implies the same thing,

according as the curve is concave or convex to the axis of x . The double sign therefore does not denote *ambiguity*, but distinguishes the results of the two hypotheses.

The double sign in [A] arises, as all such superfluous appendages must arise, from the analytical premises being capable of a wider interpretation than the geometrical conditions strictly warrant. The problem really solved by the analysis is this, namely: To determine the length of a straight line R , drawn from a point (a, β) , to a point (x, y) ; this line being perpendicular to a certain line (the tangent) through (x, y) . The analysis furnishes two points (a, β) , one on each side the tangent, as we have already explained at length elsewhere. (See *Anal. Geom.* Part I., p. 42.) The formula [B], when employed in any particular case, will involve a distinct hypothesis as to $y - \beta$, and consequently a distinct hypothesis as to $x - a$; this so qualifies the analytical conditions that the double sign before the numerator of [B] would be not merely superfluous but unjustifiable.

There is some confusion of signs in the reasoning of CAUCHY, (*App. du Calcul Infinitésimal à la Géométrie*, p. 89,) in reference to the radius of curvature. He represents the simultaneous increments of the arc s and angle α , by $\pm \Delta s$ and $\pm \Delta \alpha$; and obtains the equation $\pm \Delta s = \pm R \Delta \alpha$ (in the limit); and thus, under the appearance of the greatest generality, does really assume an hypothesis, implied in the signs prefixed to the increments; and the equation

$$\frac{1}{R} = \pm \frac{\Delta \alpha}{\Delta s} \text{ (in the limit)}$$

which he finally deduces is not in conformity to that hypothesis. The double sign should not have been prefixed to Δs , since the proper sign of it is really implied in that of $\Delta \alpha$.

(18.) If we were to take the axis of ordinates in the di-

rection of the normal at the proposed point, and consequently the axis of abscissas parallel to the tangent at that point, the radius of curvature would be obtained with remarkable simplicity. Thus: since, in the proposed hypothesis as to the axes, the abscissa a of the centre of curvature, is 0; and the radius obviously $y - \beta$, the equation of the osculating circle will be

$$x^2 + (y - \beta)^2 = R^2,$$

and, by differentiating,

$$y - \beta = - \frac{x}{\frac{dy}{dx}} = R$$

But, as at the proposed point $x = 0$, and also $\frac{dy}{dx} = 0$, to get the true value of this fraction we must differentiate both numerator and denominator: we thus obtain

$$R = - \frac{1}{\frac{d^2y}{dx^2}} \dots [D]$$

(19.) We shall merely add, in conclusion, that LACROIX is one of the very few writers who regard the double sign, usually prefixed to the expression for R , as meaningless in reference to the direction of R . But yet, in common with most others, he considers the single minus sign as arbitrarily introduced purely for convenience; whereas this sign, as we have seen, necessarily accompanies the correct result. The words of Lacroix on this subject are as follow:

“The value of R being susceptible of the double sign $\pm$, it may be asked, which of the two we ought to employ; for it is evident, that in general for one point of the curve there is but one radius of curvature; and since this radius has not, except at some particular points, the same direction with the ordinate or abscissa, it cannot properly be said to have any peculiar sign with respect to those lines. The determination therefore of that by which we

commonly affect it must depend upon a convention previously established on the direction of the curvature, with respect to the normal. If we agree to consider the radius of curvature as positive, for those curves whose concavities are turned towards the axis of the abscissæ, as the value of $\frac{d^2y}{dx^2}$ is, in these cases, negative, we must affect the expression for R with the sign —; and the same assumption will also make the radius of curvature negative when the concavities of the curves are turned from the axis of the abscissæ, since it changes its sign at the same time with $\frac{d^2y}{dx^2}$. In conformity to this convention, we shall always in the application of the formulæ, assume

$$R = - \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx d^2y}.$$

(Lacroix, *Diff. Calc.* English Trans. p. 117.)

It has been seen however that, without any assumption at all, this minus sign is a necessary part of the true result. It has no reference either to the double sign of the radical, or to the geometrical position of the radius. It refers solely to the hypothesis, before adverted to, as to $y - \beta$; an hypothesis which implies that the point under examination is in the *concave* part of the curve. If this supposition be right, the minus confirms it; if wrong, the same minus corrects it.

NOTE.

SOME advance has been of late made towards the establishing of a kind of analytical geometry, in the fundamental principles of which fewer conventions appear to enter even than in the geometry of DES CARTES. Only a single fixed axis is here employed, an axis of abscissas; along which, from a proposed origin, the only positive and

negative directions are taken ; and a directive sign, different both from + and —, although perfectly consistent with them, is then employed to mark the *positions* of lines oblique to the axis. Thus θ representing the inclination of any such oblique line drawn from the origin, the symbol $(\cos \theta + \sin \theta \sqrt{-1})$ prefixed to it will mark its direction, and will conform to the original conventions when the line by revolving comes into coincidence with the fixed axis. If the absolute length of a line from the origin be a , then $(\cos \theta + \sin \theta \sqrt{-1}) a$, in the proposed notation, represents it in length and position : if it come to coincide with the axis, taking the positive direction, then $\cos \theta = +1$, and $\sin \theta = 0$, so that the representation is $+a$; if it take the negative direction, upon coinciding with the axis, $\cos \theta$ will be -1 , and $\sin \theta = 0$, as before ; so that the proper representation will then be $-a$. If it take a direction at right angles to the axis it will be denoted by $a\sqrt{-1}$, because in that position of the line a , $\cos \theta$ will be 0, and $\sin \theta = 1$.

In this manner expressions, which in their common algebraic character denote imaginary or impossible quantities, are introduced into geometry to indicate linear direction.

Whether or not this appropriation of them is perfectly consistent with all the other conventions of analysis, it must be left for further experience to determine. The author of the present work has certainly his own objections to it ; but as it is advocated by one of the ablest of modern analysts, it must not be hastily opposed.*

* Professor PEACOCK. See Report to the Third Meeting of the British Association. Dr. Peacock has, we believe, expounded this theory at length in his *Algebra*, a work that has been long out of print, and which I have not been able to procure. The student may also consult, on this subject, Mr. Warren's papers in the *Phil. Trans.* for 1829 ; as also a work on the *Geometrical Representation of Imaginary Quantities*, published by the same gentleman in 1828.

DISSERTATION II.

ON THE CURVATURE OF SURFACES, WITH AN EXAMINATION OF THE THEORY OF UMBILICAL POINTS, AS DELIVERED BY MONGE, DUPIN, AND POISSON.

DISSERTATION II.

ON THE CURVATURE OF SURFACES.

(20.) IF we except some of the recent researches of M. CHASLES we may truly assert that to the labours of EULER, MONGE, and DUPIN we are almost exclusively indebted for a comprehensive and systematic theory of the curvature of surfaces. Our obligations to MONGE are especially due. The elegance and originality of his investigations, and the remarkable beauty of the results to which they conduct, justly entitle him to the preeminence in this difficult department of research; and render the work which he has bequeathed to posterity under the title of *Application de l'Analyse à la Géométrie*, although one of the most profound, yet at the same time one of the most attractive performances which analytical science can boast.

The *Développements de Géométrie*, by his celebrated pupil Baron DUPIN, is intended to form a supplement to the work of MONGE. It is unquestionably entitled to a place by the side of its precursor; as well on account of the consummate analytical skill which it displays, as because of the valuable contributions it furnishes to the same important object—a complete and general theory of curve surfaces.

(21.) But there is one interesting subject connected with this theory, which these illustrious authors have left in considerable obscurity: the subject of “*ombilics*,” or

those remarkable points on a surface at which the normal sections all have the same radius of curvature.

The obscurity here complained of will be found upon examination to arise in a great measure from these two causes; namely, from overlooking the change which takes place in the nature of a certain differential equation, when, by giving constant values to the coefficients, it becomes an equation of common algebra; and from the inaccurate conceptions, entertained both by MONGE and DUPIN of the import of the symbol $\frac{0}{0}$, in the analytical discussion referred to. DUPIN (*Développements*, p. 178,) clearly saw that MONGE had misinterpreted this symbol in his investigation of umbilical points; yet, as it appears to us, he himself failed to perceive its true character in the same enquiry, and is in consequence led, in the course of his reasonings, to certain positions that a close examination will prove to be untenable.

(22.) At page 160 of the *Développements*, the author, after exhibiting the general differential equation of lines of curvature, observes respecting it, that a simple inspection shows that its terms all vanish together when the two radii of curvature become equal; and that then, the equation presenting itself under the form

$$\left(\frac{dy}{dx}\right)^2 \cdot 0 + \left(\frac{dy}{dx}\right) \cdot 0 + 0 = 0,$$

$\left(\frac{dy}{dx}\right)$ becomes indeterminate, and capable of an infinite number of different values.

He remarks, however, that we ought not hastily to conclude that from the points where the two curvatures of the surface are equal, and directed in the same sense, there issue an infinity of lines of curvature. For, he adds, we know from the theory of the differential calculus that there

are cases in which certain magnitudes present themselves under an indeterminate form without being so in reality; where the general equation implies either more or less than the particular case under consideration: whence we might immediately conclude that in general, in the singular points with which we are at present occupied, there are more or fewer than two lines of curvature; there must be then only one, or three, or else a greater number; but these inductions, however excellent to guide us in the search of truth, ought never to be regarded as absolute proofs.*

(23.) Now this language evidently implies a doubt as to whether the values of $\left(\frac{dy}{dx}\right)$, which in the case before us appears under the form $\frac{0}{0}$, are really innumerable or not; and reference is made to the subsequent determination of the differential calculus for the removal of this doubt.

* “ Nous reconnâtrons, à sa seule inspection, que ses termes s'évanouissent à la fois quand les deux rayons de courbure deviennent égaux, et qu'alors cette équation s'offrant sous la forme

$$\left(\frac{dy}{dx}\right)^2 \cdot 0 + \left(\frac{dy}{dx}\right) \cdot 0 + 0 = 0,$$

$\left(\frac{dy}{dx}\right)$ devient indéterminé, et susceptible d'une infinité de valeurs différentes.

“ Cependant ne nous hâtons pas encore d'en conclure qu'aux points où les deux courbures de la surface sont égales et dirigées dans le même sens, viennent de croiser une infinité des lignes de courbure. Car nous savons par la théorie du calcul différentiel, qu'il est des cas où certaines grandeurs se montrent sous une forme indéterminée sans être telles en effet; ce sont ceux où l'équation générale dit plus ou dit moins que le cas particulier que l'on considère, d'où nous pourrions déjà conclure qu'en général, dans les points singuliers dont nous nous occupons actuellement, il y a plus ou il y a moins de deux lignes de courbure; il y en aura donc une seule, ou trois, ou un plus grand nombre; mais ces inductions excellentes pour nous guider dans la recherche de la vérité ne doivent jamais être regardées comme des preuves absolues.”

(*Développements de Géométrie*, p. 160.)

DUPIN is accordingly guided by such determination, and concludes that the values sought are not necessarily innumerable. (See *Développements*, pp. 127, 185.)

We submit, however, that in no case is the differential calculus competent to decide whether $\frac{0}{0}$, the form which a general analytical result takes in certain particular hypotheses, as to the arbitrary quantities entering that result, has, or has not, innumerable values. All that the subsequent application of the calculus can do, in such a case, is to impose upon those values, be they innumerable or not, a new condition or restriction; and to bring out from among them all those which conform to this new condition, as well as to the original restrictions implied in the fundamental equations.

The values which satisfy these equations may be innumerable; although out of these the calculus may select but one, which one however will be distinguished from all the others by the peculiarity implied in the new condition.

The calculus is not, therefore, to be resorted to in order to test the admissible values concealed under the form $\frac{0}{0}$; but only for the purpose of bringing out from among them all those which are singular or unique; those, in fact, which more than fulfil the original conditions.

It is a general analytical principle, too often overlooked certainly, but which will be assented to as soon as stated, that those values of the arbitrary quantities (and none but those) which render the equations of condition indeterminate, must also render the final result to which they lead equally indeterminate. When such a result therefore assumes the form $\frac{0}{0}$, its true character is to be tested by the equations which have led to it, after these are modified by the hypotheses from which that form has arisen. The perplexities and mistakes in the theory of umbilici, as well

as in many other analytical enquiries which terminate in ambiguous expressions, are mainly attributable to a neglect of this obvious principle.

(24.) The fundamental mistake however in the article from MONGE, criticised by DUPIN in the *Développements*, page 173, is of a different and indeed of a more remarkable kind. It consists in the supposition of an equality between the two roots of the equation of the lines of curvature, although the very form of that equation is such as to render an equality of roots absolutely impossible; an oversight which, however palpable, has escaped the detection of DUPIN and subsequent analysts. The author of the interesting paper on "Lines of Curvature," published in the fourteenth volume of the *Transactions of the Royal Irish Academy*, has fallen into the same error; and has thus given a theory of umbilical points which is fundamentally incorrect.

The equation of the lines of curvature being of the second degree, with its third term *always negative*, cannot possibly have equal roots; so that these roots, generally two, can never merge into one and coincide, although such a coincidence is frequently affirmed to be the distinguishing peculiarity of the umbilical points, and therefore the criterion of their existence upon a surface.*

It is nevertheless quite true, that at every such point, not the vertex of a surface of revolution, the lines of curvature, generally two, do really merge into one, and become

* Thus MONGE remarks, "les ombilics sont les points dans lesquels les lignes des deux espèces de courbure se changent l'une en l'autre, et par conséquent pour chacun desquels les deux lignes de courbure se confondent. Ces points sont donc ceux pour lesquels les deux valeurs de $\frac{dy}{dx}$ que fournit l'équation générale des lignes de courbure, sont égales entr'elles."—*Application de l'Analyse*, art. ix. See also LENOIR, *Analyse appliquée à la Géométrie des Trois Dimensions*, p. 308, edit. 1835.

confounded; but this uniting of the two lines of curvature in one is, as we shall endeavour to show, a thing quite distinct from the confounding of the two roots of the quadratic equation above adverted to, when this equation is rendered determinate by substituting in it, for x and y , the constant coordinates of the umbilicus.

Could such a coincidence of the roots take place, we should be presented with a very remarkable circumstance, one altogether unique in analysis, an instance of two quantities becoming equal in a certain extreme hypothesis, as to the disposable magnitudes, without their making the least approach to such equality during the progress of those magnitudes towards the ultimate state.

(25.) In a *Mémoire sur la Courbure des Surfaces*, published in 1832 in the *Journal de l'Ecole Polytechnique*, tome xiii., POISSON has considered the disputed question as to the number of lines of curvature passing through an umbilical point; and he has arrived at the conclusion, that these lines are infinite in number, and issue from the umbilicus in all directions; and that those particular lines, selected from among these by DUPIN, differ from the others only by their satisfying an additional analytical condition. But this is not the correct theory. The conclusions of DUPIN on this subject, although we think some of his reasonings to be not only obscure but unsound, and shall accordingly venture to animadvert on them in the course of this dissertation, do really involve the true doctrine of the lines of curvature; and those of these lines, which DUPIN has shown to issue from an umbilicus, are not selections from an infinite number of such lines, but do really comprise all that can possibly proceed from that point.

Moreover, the mode of investigation pursued by POISSON, in order to detect from among the infinite number of lines which he affirms to pass through an umbilicus those dis-

tinguished by DUPIN, is peculiar; and, as appears to us, not only far-fetched, but wanting in that analytical elegance which generally marked the researches of that eminent philosopher.

His professed object seems to have been to reconcile the conflicting doctrines of MONGE and DUPIN on the subject of umbilici; the former appearing to think that an infinite number of lines of curvature might proceed from these points, and DUPIN distinctly maintaining, that their number must always be limited, except at a vertex of revolution: he agrees with MONGE that the number is infinite, and so far with DUPIN that a limited number of them are distinguished by peculiarities. But even supposing this to be the true theory, which it is not, his analytical process is so entirely different from theirs, that it could be of no assistance whatever towards removing the obscurities and discrepancies of his predecessors, the causes of which should be sought, not in their conclusions, but in the reasonings which have led to them. We suspect that the novel mode of investigation employed by POISSON was adopted by him only because of his concurring with DUPIN in the opinion, that at the umbilical points "*la méthode ordinaire pour trouver les lignes de courbure est en défaut.*" (*Développements*, page 359.)

(26.) POISSON, in common with all subsequent writers, infers, that when the coefficients of the equation which determines the direction of curvature all vanish, the infinite number of roots of that equation are all admissible, an inference not necessarily true; and one which, as we have seen from the quotation at page 35, DUPIN was careful to avoid. Passing by this, however, it does not clearly appear from POISSON's investigation, nor from any that has hitherto been given, what the *geometrical* peculiarity of the surface in the directions of DUPIN's lines of curvature at an umbilicus really is. These *directions* are certainly dis-

tinguished from all other directions from the point by peculiar analytical conditions: to what geometrical circumstances do these refer? Poisson, indeed, replies, that the approximation of the consecutive normals, in the directions which the analysis points out, is infinitely more close than in any other direction; a statement which, although it really does involve an interesting geometrical fact, we cannot but regard as too vague to convey any satisfactory idea as to the peculiarity of the surface in the directions referred to.

In the present dissertation it will be our endeavour to remove the obscurities connected with this subject, and at the same time to develop some of the more important properties of curve surfaces in general; adopting a mode of investigation somewhat different from that usually employed in such enquiries, and which will conduct us to those properties by a very short and easy route.

(27.) The general equation of any surface is

$$Z = F(X, Y) \dots [1]$$

Let the coordinates X, Y, Z , take the increments x, y, z ; then we shall have for z the value

$$z = \frac{dZ}{dX} x + \frac{dZ}{dY} y + \frac{d^2Z}{dX^2} \frac{x^2}{1.2} + \frac{d^2Z}{dXdY} xy + \frac{d^2Z}{dY^2} \frac{y^2}{1.2} + \&c.$$

This equation may be equally regarded as the representation of the surface [1]; the coordinates x, y, z , of any point in it, being now measured from the point X, Y, Z , on the surface, as the origin; the axes in the one case being parallel to those in the other.

If we suppose the axes to be rectangular, and that of z to coincide with a normal to the surface, the axes of x and y will be in the tangent plane; and, under this arrangement,

the coefficients of x and y must vanish, so that the equation of the surface becomes simply

$$z = \frac{d^2Z}{dX^2} \frac{x^2}{1.2} + \frac{d^2Z}{dXdY} xy + \frac{d^2Z}{dY^2} \frac{y^2}{1.2} + \&c. \dots [1]$$

When z is a rational and integral function of x and y , the series on the right terminates with those quantities having the same dimensions as the proposed equation. But when z is not a rational and integral function of x and y , the series extends indefinitely, the higher powers of x and y arising merely from developing the irrational part of the expression for z . When the equation is of the second order, the terms beyond $\frac{y^2}{1.2}$ obviously represent merely the function $Pz^2 + Qxz + Ryz$.

The equation of any plane, parallel to that of xy , and at the distance h from the origin, will be

$$z = h \dots [2]$$

Hence, for the curve of intersection made by this plane with the surface, we have the equation

$$\frac{d^2Z}{dX^2} \frac{x^2}{2} + \frac{d^2Z}{dXdY} xy + \frac{d^2Z}{dY^2} \frac{y^2}{2} + \&c. = h;$$

and, consequently, the equation of the curve of the second order to which this indefinitely approaches as h (or z) diminishes; that is, as the plane of intersection approaches towards the tangent plane, is

$$\frac{d^2Z}{dX^2} \frac{x^2}{2} + \frac{d^2Z}{dXdY} xy + \frac{d^2Z}{dY^2} \frac{y^2}{2} = h \dots [3]$$

which equation represents in general only a point, namely, the point of contact, when the moving plane [2] actually reaches the tangent plane; that is, when h , and conse-

quently x and y , become zero. Nevertheless, before the moving plane attains this ultimate position, the section made by it with the surface will obviously be a curve approaching nearer and nearer to coincidence with the curve [3] as the plane advances towards coincidence with the tangent.

(28.) If the surface under consideration were itself of the second order, the form of our general equation would then be

$$z = \frac{d^2Z}{dX^2} \frac{x}{2} + \frac{d^2Z}{dXdY} xy + \frac{d^2Z}{dY^2} \frac{y^2}{2} + z(pz + qx + ny) \dots [4]$$

as already remarked; and the curve of intersection would be

$$\frac{d^2Z}{dX^2} \frac{x^2}{2} + \frac{d^2Z}{dXdY} xy + \frac{d^2Z}{dY^2} \frac{y^2}{2} + h(pz + qx + ny) = h \dots [5]$$

which curve, we know, always continues similar to itself during the progress of the intersecting plane, (*Anal. Geom.* Part II. p. 210,) however much h may diminish. The similarity being thus undisturbed by the variation of h , continues then up to the limit; so that when this limit is actually reached, and h becomes zero, whatever ratio any two diameters may have all along preserved, in that same ratio must they vanish when h , x , y , become zero, or when the section becomes a point; the same point, in fact, as that on the original surface, and represented above by [3], when $h = 0$.

This ultimate section however is not necessarily always a point; whether it be or not will depend upon the relation among the three constant coefficients. If this relation be such that

$$\left(\frac{d^2Z}{dXdY} \right)^2 = \frac{d^2Z}{dX^2} \cdot \frac{d^2Z}{dY^2}$$

the contact of the tangent plane will be, not a point merely, but a straight line, (*Anal. Geom.*, Part II. p. 26;) and if the relation be

$$\left(\frac{d^2Z}{dXdY}\right)^2 > \frac{d^2Z}{dX^2} \cdot \frac{d^2Z}{dY^2}$$

the surface will be touched by the plane along two straight lines which cross at the point under consideration, (*Anal. Geom.* p. 20.) These particulars will be otherwise established at art. (39.)

(29.) Such ultimate intersections as these will imply that the preceding sections [5], which have finally merged into them, are all parabolas or hyperbolas, and that the surface is not uniformly convex or concave about the point under examination; nevertheless, agreeably to what has already been shown, whatever diameters or chords of the similar curves of the parallel sections of the surface [4], may have vanished at this limit, they must have vanished in the ratio that they all along had. So far therefore as such ratios only are concerned, we may infer the properties of the evanescent curve, or tangential section [3] of our proposed surface, h being zero, by discussing the properties of the finite or real curve [5], h being any finite value; or, which is the same thing, we may speak of the diameters, &c., of the evanescent curve, as far as their ratios are concerned, just as if these diameters were really of finite length, and not so many zeros.

(30.) This evanescent curve, the limit of the intersections of the plane [2] with the proposed surface, is that which DUPIN calls the *indicatrix* of the surface at the point X, Y, Z . Its equation, when agreeably to the usual practice r, s, t is put for the three differential coefficients in [3], will be

$$rx^2 + 2sxy + ty^2 = 2h \dots [6]$$

understanding of course that the relations among x , y , and h , are, for this indicatrix, those which these lines have in the limit.

If now any normal section through our point X , Y , Z be made, h or z will represent, what may be called, the versed sine of the section; when h becomes evanescent the corresponding sine will be that semidiameter of the indicatrix through which the section passes. The ratio of the square of this evanescent sine to the corresponding versed sine is a common expression for the radius of curvature of the section at the proposed point, an expression immediately deducible from EUCLID, (Prop. 36, Book III.,) in conjunction with the doctrine of limits. This radius, however, being an important element in the theory of curvature, we shall establish its value in a manner more strictly analytical.

(31.) We have already seen (page 28) that the absolute length of the radius of curvature at a point in a plane curve at which the tangent is parallel to the axis of abscissas, and which axis we may, for the moment, call the axis of x is

$$R = \frac{1}{\frac{d^2y}{dx^3}}$$

and since in this case $y = 0$, and $\frac{dy}{dx} = 0$, the ordinate h , of the curve, due to an abscissa a , is

$$h = \frac{1}{2} \frac{d^2y}{dx^2} a^2 + \dots$$

The ratio of this ordinate to a^2 is

$$\frac{h}{a^2} = \frac{1}{2} \frac{d^2y}{dx^2} + \dots = \frac{1}{2} \frac{d^2y}{dx^2}, \text{ in the limit.}$$

(32.) Hence, calling the semidiameter of the indicatrix, through which a normal section passes, a , we have

$$R = \frac{a^2}{2h} \dots\dots [7]$$

In like manner for the radius of curvature R' , of a normal section through another semidiameter a' , we have

$$R' = \frac{a'^2}{2h} \therefore \frac{R}{R'} = \frac{a^2}{a'^2}$$

so that the radii of curvature of normal sections at any point are to each other as the squares of those diameters of the indicatrix through which the sections pass.

(33.) If a, a' be the principal semidiameters of the indicatrix, and a'' any other semidiameter, then (*Anal. Geom.*, pp. 104, 150,)

$$\frac{1}{a''^2} = \frac{\sin^2 \alpha}{a^2} + \frac{\cos^2 \alpha}{a'^2}$$

where α is the inclination of a'' to a . Hence [7] calling the radius of curvature of the section through a'' , R'' , we have the well-known property of EULER, namely,

$$\frac{1}{R''} = \frac{1}{R} \sin^2 \alpha + \frac{1}{R'} \cos^2 \alpha \dots\dots [8]$$

showing that the curvature of any normal section at a point is equal to the sum of the principal curvatures at that point multiplied respectively by the squares of the cosines of the inclinations of the proposed section to the principal sections.

Since

$$\sin^2 \alpha = \frac{1}{2} - \frac{1}{2} \cos 2\alpha, \text{ and } \cos^2 \alpha = \frac{1}{2} + \frac{1}{2} \cos 2\alpha$$

the above expression may be written in the form

$$R'' = \frac{2 R R'}{R' + R - (R' - R) \cos 2\alpha}.$$

(34.) If, instead of the principal semidiameters, we take a, a' , to represent any rectangular semidiameters whatever, then, as it is a general property of the conic sections that

$$\frac{1}{a^2} + \frac{1}{a'^2} = \text{constant},$$

it follows that

$$\frac{1}{R} + \frac{1}{R'} = \text{constant},$$

that is, the algebraic sum of the curvatures of any two normal sections, at right angles to one another, is a constant quantity; observing that $\frac{1}{R'}$, will take a positive or a negative sign according as a, a' belong to an ellipse or to an hyperbola. That the sections of greatest and least curvature are always at right angles to one another follows immediately from the fact that the greatest and least diameters are at right angles to one another.

(35.) If a, a' be any pair of conjugate semidiameters then

$$a^2 + a'^2 = \text{constant},$$

$$\therefore R + R' = \text{constant};$$

so that the sum of the radii of curvature of the normal sections passing through a pair of *conjugate tangents* to the surface at any point is a constant quantity, namely, the sum of the principal radii, or the radii of curvature of the surface at that point.

(36.) The radii of curvature of the sections through the *equal* conjugate diameters being themselves equal, it follows that each of these is an arithmetical mean between the two radii of the surface. Such sections therefore might with propriety be called the *sections of mean curvature* of the surface at the point: a sphere whose radius is that of a section of mean curvature, would not have contact of the second order with the *surface* in the directions of these mean sections, yet the sections themselves would osculate, and the sphere would represent the mean or average curvature of the surface at the proposed point. It might be called the *sphere of average curvature* at that point; and when the curvature is uniform, this sphere would be the osculating sphere.

(37.) As respects the normal sections, we see that h remains the same for each, while a , that semidiameter of the indicatrix through which the section passes, differs with the direction of a ; that is, these semidiameters are not in a ratio of equality. But if through any proposed semidiameter *oblique* sections be made to pass, then, obviously, the h in those sections would differ, whilst the a remains the same for all. In fact, if ϕ were the inclination of any oblique section through a , to the normal section through the same, we should evidently have the relation $h' = \frac{h}{\cos \phi}$, calling the oblique h , h' . Hence the radius ρ of the oblique section would be

$$\rho = \frac{a^2}{2h'} = \frac{a^2}{2h} \cos \phi = R \cos \phi$$

which is the theorem of MEUSNIER; and from which we see that the radius of an oblique section is always a shorter line than the radius of the normal section through the same linear tangent. But a more interesting property is

immediately deducible from this theorem, as will appear from the following considerations :

(38.) It is plain that two surfaces must mutually osculate at a point when the normal sections all osculate at that point. Now in order to this it is only necessary that the principal sections be in the same normal planes, and have respectively the same radii of curvature. For the indicatrix at the point of contact must then have its principal diameters the same for one surface as for the other, so that the same indicatrix must be common to both ; and therefore a normal plane through any diameter, $2a$, of the indicatrix must give sections, having a common radius, namely, $\frac{a^2}{2h}$. Such being the case it follows from the theorem of MEUSNIER that

If two surfaces touch at a point, and their principal sections lie in the same planes, and have respectively the same radii of curvature, then whatever plane be drawn through the point, whether normal or oblique, the two sections made by it will mutually osculate at that point.

(39.) It is useful to express the value of R , the radius of any normal section, in terms of the inclination θ of the trace of that section to the axis of x . To obtain such an expression substitute for a^2 in [7] its value in terms of x', y' the coordinates of the extremity of that semidiameter: we shall thus have

$$\begin{aligned} R &= \frac{x'^2 + y'^2}{2h} \\ &= \frac{1 + \frac{y'^2}{x'^2}}{r + 2s \frac{y'}{x'} + t \frac{y'^2}{x'^2}} \\ &= \frac{1 + \tan^2 \theta}{r + 2s \tan \theta + t \tan^2 \theta} \dots\dots [8] \end{aligned}$$

agreeing with the common expression usually deduced by more operose methods.

If for $1 + \tan^2 \theta$ we put its value $\sec^2 \theta$, and then divide numerator and denominator by this, the form for R will be

$$R = \frac{1}{r \cos^2 \theta + 2s \sin \theta \cos \theta + t \sin^2 \theta} \dots\dots [9]$$

If it be possible to draw a straight line upon the surface through the point under examination, then for the section through this line we ought to have $R = \infty$; and consequently from [8]

$$r + 2s \tan \theta + t \tan^2 \theta = 0 \dots\dots [10]$$

so that the direction in which such line lies is

$$\tan \theta = \frac{\sqrt{s^2 - tr} - s}{t};$$

Hence, that only one straight line may pass through the point and lie upon the surface, there must exist the condition already adverted to at page 42, namely,

$$s^2 - tr = 0,$$

and in order that two straight lines through the point may lie upon the surface, we must have

$$s^2 - tr = +,$$

a condition which must always have place when, at the proposed point, the normal plane by turning round passes from convex sections to concave, or vice versa; the two intersecting lines being the boundaries which separate the convex compartments around the point from the concave.

(40.) When the former of these two conditions holds, the indicatrix is a parabola, or rather that extreme variety

of this curve which takes the form of two parallel straight lines, ultimately merging into one (See *Anal. Geom.* Part. II., p. 25.) The parallel, midway between these, is the line of centres; that is, every point in this line, bisecting all the chords through it, may be regarded as a centre; and each chord a diameter; the line of centres and the perpendicular chord through any centre being the principal diameters in reference to that centre. Calling the finite principal semidiameter a , any other semidiameter a'' , inclined to it at an angle α , will obviously be $a'' = a \frac{1}{\cos \alpha}$; consequently

$$\frac{1}{a'^2} = \frac{1}{a^2} \cos^2 \alpha \quad \therefore \frac{1}{R''} = \frac{1}{R} \cos^2 \alpha,$$

an inference that we might at once have obtained from EULER'S formula [8]; since the diameter a' , at right angles to α , being infinite, $\frac{1}{a'^2}$, and consequently $\frac{1}{R'}$, is zero.

From the expression just deduced we have for the radius R'' of any oblique section, in terms of the radius of the section perpendicular to the straight line, and the inclination between the two, the value

$$R'' = R \sec^2 \alpha.$$

(41.) When the latter of the two conditions above has place, the indicatrix is a hyperbola, having the intersecting straight lines through the point for its asymptotes. The directions of maximum and minimum curvature, that is, of the principal diameters of this hyperbola, will therefore lie in the lines bisecting the adjacent angles formed by these asymptotes. As the squares of the principal diameters $2a$, $2a'$, have, in this case, opposite signs, we infer that the two principal sections have curvatures of opposite

kinds; and the same may be said of every pair of sections through conjugate diameters: the extreme pair of these diameters, the asymptotes, give sections neither convex nor concave, but rectilinear.

(42.) It may be useful to observe here, that it is because of the change of curvature that takes place at these rectilinear sections that they are not regarded as the sections of minimum curvature. For although in reality the curvature of every such section is $\frac{1}{R} = 0$, yet this value, under the circumstances, does not fulfil the condition of a maximum or minimum. In developable surfaces however, for which the relation $s - tr = 0$ everywhere holds, the rectilinear section through any point *does* fulfil the analytical condition of a minimum, since the curvature upon both sides of such a section is of the same kind; so that $\frac{1}{R}$, in passing from one side to the other, through the value 0, undergoes no change of sign.

(43.) If in the expression [8] we put $s = 0$ and $r = t$, then

$$R = \frac{1 + \tan^2 \theta}{r(1 + \tan^2 \theta)} = \frac{1}{r},$$

so that at the point where those conditions have place, the radius of curvature is constant for every normal section. This point, therefore, is an "ombilic," and the indicatrix a circle.

(44.) Thus far we have limited our examination of the surface to an isolated point taken upon it; and have deduced the more important particulars respecting the curvature of the surface there. For such an examination the indicatrix, as we have seen, offers great facilities; as it

readily leads to the correct expressions for the radii of the normal and oblique sections, and thus *indicates* the character of the surface at any proposed point. The student ought, however, to be reminded that, however complicated be the surface, the indicatrix, from the properties of which our several deductions have been obtained, is an evanescent curve of only the *second order*; and that, consequently, it is not rigidly the actual curve in which the series of parallel sections, made in a surface of a high order, vanishes when the tangent plane is reached; for strictly speaking this will generally be a curve of the same order as the surface; although to no curve of the *second order* does it so closely approach as to that which is called the indicatrix. This approximate indicatrix however is all that is necessary for the accurate determination of the character of any surface at a proposed point; so that any correction of it, by introducing into its equation terms of a higher order, would, for the purpose to which it is applied, be altogether useless. The expression [7], from which the subsequent ones have mainly been deduced, involves only the semidiameter a of the indicatrix of the second order:—a fact which deserves especial notice.

When the surface under consideration is itself of the second order our indicatrix is, of course, rigidly correct; we shall briefly consider it, in reference to these surfaces, before proceeding to discuss the lines of curvature of surfaces in general.

(45.) A surface of the second order osculates any other surface when the indicatrix of the former, at the point of contact, is also the indicatrix of the second order of the other surface; as is plain from what has been proved at (38), or from the equations [1] and [4]. (See *Diff. Calc.*, p. 225.) We may therefore say that two surfaces osculate at a point when they have a common indicatrix there. Taking the axes of x, y , and z as before, we have for two

surfaces of the second order, osculating at the origin, the equations

$$z = ax^2 + bxy + cy^2 + mx^2 + nxz + pyz$$

$$z = ax^2 + bxy + cy^2 + m'x^2 + n'xz + p'yz,$$

the equation of the common indicatrix being

$$ax^2 + bxy + cy^2 = 0.$$

If, however, the axes of x and y be made to coincide with the principal diameters of the indicatrix, then b in this equation must be zero; so that the two osculating surfaces may be rather more simply represented by the equations

$$z = ax^2 + cy^2 + mx^2 + nxz + pyz \dots\dots[11]$$

$$z = ax^2 + cy^2 + m'x^2 + n'xz + p'yz \dots\dots[12].$$

The equation of the common indicatrix being

$$ax^2 + cy^2 = 0,$$

it will be a point provided a and c have the same sign, a pair of intersecting straight lines if they have opposite signs, and only a single straight line if one of these be zero. If $a = c$ the surfaces osculate at an umbilical point.

(46.) Besides the indicatrix, the surfaces may have other points in common; these will be discovered by determining all the values of the coordinates for which the equations [11], [12] exist simultaneously. In order to this subtract [12] from [11] and we shall have

$$(m - m') z^2 + (n - n') xz + (p - p') yz = 0,$$

an equation which resolves itself into the two

$$z = 0 \text{ and } (m - m') z + (n - n') x + (p - p') y = 0.$$

The first has already been considered; it represents the

indicatrix; the second represents a plane passing through the point of osculation. This plane however coincides with the tangent plane if $n = n'$ and $p = p'$; in which case therefore the surfaces are entirely separated, except at their point of osculation.

The two surfaces [11], [12], so long as this equality has not place, have contact of the second order only; for contact of the first order requires merely that, in the developments of z , the terms containing the first powers of x , y , be identical: contact of the second order requires, moreover, that the terms containing the second powers of these should also be the same in the two developments. (See *Diff. Calc.* p. 225.) These conditions are fulfilled by the common indicatrix. But, if $n = n'$, and $p = p'$, the contact will be of the third order; because then, in the developments spoken of, the third powers of x and y , or the terms involving these in three dimensions, would be the same in both. This is plain; for, putting the term containing x^3 last, the equations may be written

$$z = ax^2 + cy^2 + (nx + py)(ax^2 + cy^2 + \dots) + mx^2$$

$$z = ax^2 + cy^2 + (n'x + p'y)(ax^2 + cy^2 + \dots) + m'x^2;$$

which, when $n = n'$, and $p = p'$, become identical, as far as terms in x and y of the third order are concerned.

It follows, therefore, that the equations

$$z = ax^2 + cy^2 + mx^2 + nxz + pyz$$

$$z = ax^2 + cy^2 + m'x^2 + nxz + pyz$$

represent surfaces having contact of the third order, and that such surfaces cannot intersect as well as touch. If n and p be each zero, the point of osculation will be the common vertex of both surfaces.

(47.) If in the equation [12], m' , n' , and p' , be each zero, the surface will be a paraboloid, whose vertex is the point

of osculation with the other surface. The equation of the plane of the curve in which the surfaces then cut is

$$mx + nx + py = 0 \dots [13]$$

$$\therefore z = -\frac{n}{m}x - \frac{p}{m}y.$$

Substituting this value of z in the equation of the paraboloid, we have for the projection of the curve of intersection upon the tangent plane, the equation

$$ax^2 + cy^2 + \frac{n}{m}x + \frac{p}{m}y = 0.$$

The coordinates of the centre of this projected curve are

$$a = \frac{n}{2am}, \beta = \frac{p}{2cm};$$

and consequently, when the curve is referred to its axes and centre, by substituting for x, y , in the above equation, the values

$$x + \frac{n}{2am}, y + \frac{p}{2cm},$$

the equation is

$$ax^2 + cy^2 = \frac{1}{4} \left(\frac{n^2}{am^2} + \frac{p^2}{cm^2} \right).$$

Hence the curve of intersection is a similar curve to the indicatrix, or to the sections parallel to the tangent plane,—a remarkable property of the osculating paraboloid.

(48.) This theorem is investigated nearly as above by M. OLIVIER in the *Journal de Mathématiques* for May 1838, who gives it as a new property of the paraboloid. But it was established long before by DUPIN in his *Développements*, and indeed in a form still more general. It may be deduced in this form, as follows:

Let a, b, c , be the coordinates of any proposed point

upon a curve surface, the axes of coordinates being rectangular, but in other respects arbitrary: then, X, Y, Z , being the coordinates of any point near the former, in passing from one to the other, the increments of the coordinates will be $X - a, Y - b, Z - c$; and the development of the latter increment in powers of the former will, as at p. 40, be

$$Z - c = p(X - a) + q(Y - b) + \frac{1}{2}r(X - a)^2 \\ + s(X - a)(Y - b) + \frac{1}{2}t(Y - b)^2 + \&c. \dots [14],$$

where p, q, r, s, t , &c. represent the values of the several partial differential coefficients of Z for the point (a, b, c) .

The equation of a tangent plane through (a, b, c) is

$$Z - c = p(X - a) + q(Y - b) \dots [15].$$

A plane parallel to this, and passing through a point on c prolonged, so as to become $c + h$, will therefore be represented by

$$Z - c - h = p(X - a) + q(Y - b) \dots [16];$$

and as [14] and [16] exist simultaneously for the points X, Y, Z , of the surface, and the curve of intersection with the plane, we have, by subtraction, the equation

$$h = \frac{1}{2}r(X - a)^2 + s(X - a)(Y - b) + \frac{1}{2}t(Y - b)^2 + \dots$$

for the projection of this curve on the plane of XY , which curve becomes the indicatrix of the surface at the point (a, b, c) , when h , by the approach of our secant [16] to the tangent [15], becomes evanescent. Hence the projection of the indicatrix of the second order has for equation

$$r(X - a)^2 + 2s(X - a)(Y - b) + t(Y - b)^2 = 2h;$$

h being zero; and this is the projection of the complete

indicatrix when the surface is itself of the second order, and h may then be of any finite value, without affecting the character of the curve.

(49.) The general equation of the paraboloid, when the coordinates originate at the vertex, and have their planes coincident with the principal planes of the surface is (*Anal. Geom.* Part II. p. 181.)

$$\frac{X^2}{P'} + \frac{Y^2}{P} = Z,$$

where P, P' are the parameters of the parabolic sections made by the planes of YZ, XZ .

Now for every point (a, b, c) , on the surface, we have, as appears by taking the differentials of the second order,

$$r = \frac{2}{P'}, s = 0, t = \frac{2}{P}.$$

Hence the equation of the projection on the tangent plane through the vertex, of the indicatrix at any point (a, b, c) , whatever on the surface, is by last article,

$$\frac{1}{P'} (X - a)^2 + \frac{1}{P} (Y - b)^2 = h;$$

which obviously becomes the same as the indicatrix at the vertex itself, by simply removing the origin of X and Y to the point (a, b) , on the tangent plane: the two curves, therefore, are similar and similarly placed on the tangent plane through the vertex; and as, in surfaces of the second order, parallel sections always give similar and similarly placed curves, we infer generally with DUPIN, that

All the plane sections of a paraboloid, projected upon a plane perpendicular to the axis, are similar curves, and similarly placed, to that given by the perpendicular plane itself. It is clear, that if the cylinders which project these sections be cut by any oblique plane, similar sections will also result, so that the sections of the surface will always

be projected into similar and similarly situated curves, whatever be the plane of projection, provided only the projecting lines be parallel to the axis of the surface.

The property at p. 55 is obviously only a particular case of this general theorem.

If the paraboloid be one of revolution, then the projection of any section parallel to a tangent plane, upon the plane perpendicular to the axis, will be a circle. But it may be remarked that, whether the paraboloid be of revolution or not, in the extreme case, where the tangent plane is at a point on the surface infinitely remote from the vertex, the sections parallel to it would seem to be straight lines, when projected upon a plane perpendicular to the axis. Every such line will, however, only be the ultimate form of a series of similar curves, whose axes, preserving a constant ratio, continually increase, till they finally become infinite in length.

(50.) When a paraboloid osculates a surface of revolution of the second order, at the vertex of the latter, the two can never intersect. For since in the equation [11] of a surface of revolution, both n and p must be zero, the equation [13] of the plane of intersection would be $mz = 0$, or $z = 0$; which represents merely the tangent plane at the common vertex. Hence the sphere, osculating the paraboloid of revolution at its vertex, will lie wholly within the parabolic surface, and so also will an osculating ellipsoid. But the hyperboloid of revolution which is osculated at its vertex by a paraboloid, will wholly comprehend the latter surface. This is obvious, because the tangent plane to the paraboloid approaches nearer and nearer to parallelism with the axis of z as the point of contact becomes more remote; so that if the parabolic surface could, around the common vertex, lie *without* the hyperbolic, a tangent plane to the former would at length cut the latter, and, *à fortiori*, the intermediate surface would cut it.

(51.) Let us now resume the consideration of surfaces in general, for the purpose of determining their lines of curvature, or those curves at every point in each of which a normal to the surface is intersected by a consecutive normal in the direction of that curve.

Let

$$\left. \begin{array}{l} A = 0 \\ B = 0 \end{array} \right\} \dots [17]$$

be the equations of a normal at any point (x, y, z) , of the surface $z = f(x, y)$, the axes being arbitrary. The equations of a second normal through $(x + \Delta x, y + \Delta y, z + \Delta z)$ will be

$$\left. \begin{array}{l} A + \frac{dA}{dx} \Delta x + \frac{dA}{dy} \Delta y + \&c. = 0 \\ B + \frac{dB}{dx} \Delta x + \frac{dB}{dy} \Delta y + \&c. = 0 \end{array} \right\} \dots [18];$$

and if these intersect in a point (x', y', z') , the equations [17], [18], must exist simultaneously. As these equations are four in number, the three quantities, x', y', z' , involved in A and B, may be eliminated from them, leaving for the result a single equation, involving, besides the fixed coordinates x, y, z , of the proposed point, the increments $\Delta x, \Delta y$. These increments therefore always have a dependence upon each other; which dependence, when the normals are consecutive, will mark out the direction from (x, y, z) , in which the consecutive intersections takes place. Foreseeing, then, that the increments are dependent, the four equations above, when subjected to the restrictions necessary for consecutive intersection, reduce to

$$\left. \begin{array}{l} \frac{dA}{dx} + \frac{dA}{dy} \cdot \frac{dy}{dx} = 0 \\ \frac{dB}{dx} + \frac{dB}{dy} \cdot \frac{dy}{dx} = 0 \end{array} \right\} \dots [19];$$

or, substituting for A and B their known values, the equations of condition are

$$1 + p(p + q \frac{dy}{dx}) + (z - z')(r + s \frac{dy}{dx}) = 0 \dots [20]$$

$$\frac{dy}{dx} + q(p + q \frac{dy}{dx}) + (z - z')(s + t \frac{dy}{dx}) = 0 \dots [21].$$

These equations, however, are not immediately fit for determining $\frac{dy}{dx}$, seeing that the indeterminate quantity z' , which is actually dependent upon $\frac{dy}{dx}$, still enters them. Eliminating this, therefore, we have finally

$$\left\{ \frac{1+q^2}{t} - \frac{pq}{s} \right\} st \frac{dy^2}{dx^2} + \left\{ \frac{1+q^2}{t} - \frac{1+p^2}{r} \right\} rt \frac{dy}{dx} - \left\{ \frac{1+p^2}{r} - \frac{pq}{s} \right\} rs = 0 \dots [22];$$

an equation of the second degree, showing that, ordinarily, there are lines of curvature proceeding from the same point in only two distinct directions.

(52.) Suppose the point to be such that for the coordinates of it there exists the double equation

$$\frac{1+p^2}{r} = \frac{pq}{s} = \frac{1+q^2}{t} \dots [23];$$

the equation [22] is then satisfied with *any* value for $\frac{dy}{dx}$, and the inference from this usually is, that under these conditions, lines of curvature issue from the point in every direction. But this conclusion is at least premature,

because the innumerable values of $\frac{dy}{dx}$, given by the equation [22], are not all admissible, unless in the proposed hypothesis the conditional equations themselves are equally indeterminate. It is necessary, therefore, to examine these before the above conclusion can be legitimately drawn, since we know that the indeterminate form $\frac{0}{0}$ sometimes arises from the introduction of foreign factors giving solutions which are inconsistent with the premises. That our conditional equations [20] and [21] may be really indeterminate, their coefficients must satisfy the conditions

$$\left. \begin{aligned} 1 + p^2 + (z - z')r &= 0 \\ pq + (z - z')s &= 0 \end{aligned} \right\}$$

and

$$\left. \begin{aligned} 1 + q^2 + (z - z')t &= 0 \\ pq + (z - z')s &= 0 \end{aligned} \right\}$$

and these evidently resolve themselves into the double condition [23].

We infer, therefore, that at the proposed point the directions, $\frac{dy}{dx}$, in which the consecutive normals intersect are innumerable: they intersect in every direction.

(53.) It is impossible that, the values of $\frac{dy}{dx}$, ordinarily two in number, can ever coincide and become one; although this is affirmed by MONGE, DUPIN, and more recent authors; for by placing the rectangular coordinate planes, which have hitherto been arbitrary, so that the

plane of xy may be either coincident with, or parallel to the tangent plane at the point under consideration, p and q will both be 0; and the equation [22] will consequently become

$$\frac{dy^2}{dx^2} + \frac{r-t}{s} \frac{dy}{dx} - 1 = 0 \dots\dots [24]$$

which can never have equal roots, since the absolute term is negative; nor can the roots ever be imaginary, for a like reason. As under this arrangement of the coordinate planes the conditions [23] become identical with those (page 51) which characterize an umbilical point—since it is impossible that p and q can be 0 in [23] without s being so too—it follows that from such a point the consecutive normals intersect in every direction; yet whatever single direction may be fixed upon either capriciously, or because of any peculiarity in that direction; in other words, whatever particular value be given to $\frac{0}{0}$, the coefficient of $\frac{dy}{dx}$ in [24], a second direction, at right angles to that chosen, is always furnished by the same equation.

(54.) That some of these innumerable directions may be distinguished from the others by certain peculiarities is highly probable; they may for instance be directions of closest contact with the osculating sphere; the question arises then, what is the analytical character of such particular directions, or of the particular lines of curvature issuing from the point in these directions?

In order to answer this question let us take a normal in the vicinity of the umbilical point. Its equations obtained by changing x, y into $x + \Delta x, y + \Delta y$ in [17] may be written in the following form, h being put without the braces for Δx ,

$$\left. \begin{aligned} & A + \left\{ \frac{dA}{dx} + \frac{dA}{dy} \frac{\Delta y}{\Delta x} \right\} h + \\ & \frac{1}{2} \left\{ \frac{d^2 A}{dx^2} + 2 \frac{d^2 A}{dx dy} \frac{\Delta y}{\Delta x} + \frac{d^2 A}{dy^2} \left(\frac{\Delta y}{\Delta x} \right)^2 \right\} h^2 + \&c. = 0 \\ & B + \left\{ \frac{dB}{dx} + \frac{dB}{dy} \frac{\Delta y}{\Delta x} \right\} h + \\ & \frac{1}{2} \left\{ \frac{d^2 B}{dx^2} + 2 \frac{d^2 B}{dx dy} \frac{\Delta y}{\Delta x} + \frac{d^2 B}{dy^2} \left(\frac{\Delta y}{\Delta x} \right)^2 \right\} h^2 + \&c. = 0. \end{aligned} \right\} \dots [25]$$

Now this normal approaches nearer and nearer to that through the umbilicus, as h diminishes; and the point in which they intersect approaches nearer and nearer to the centre of the osculating sphere. This point is actually reached only when h becomes zero, and the normals coincide; and we have already seen that a necessary condition accompanying this coincidence of the intersecting normals is that the coefficient of h in both equations, as well as h itself, must become zero; that is, we must have

$$\left. \begin{aligned} \frac{dA}{dx} + \frac{dA}{dy} \frac{dy}{dx} &= 0 \\ \frac{dB}{dx} + \frac{dB}{dy} \frac{dy}{dx} &= 0 \end{aligned} \right\} \dots\dots [26]$$

There is no necessity that the next coefficients, those of h^2 , should also become zero in the ultimate position of the moving normal, in order that it may intersect the fixed one at the centre of the osculating sphere. It is sufficient that all the subsequent terms, of the form $Ph^3 + Qh^4 + \&c.$, tend towards and become zero when that position is reached; and this they do simply by the diminution and ultimate evanescence of h , though P , Q , &c., should continue finite. Nevertheless, the more of these coefficients which tend to zero and vanish with h , the more nearly will the moving normal, for a small value of h , approach to the

centre of the osculating sphere, the point of ultimate intersection; inasmuch as the terms $Ph^2 + Qh^3 + \&c.$, for such small value of h , are then more nearly evanescent than under other circumstances; and although when we say that two normals, be they ever so close together, actually cross at the centre of curvature, we commit error, yet it is plain that that error is necessarily diminished more and more the nearer the coefficients of $h^2, h^3, \&c.$, approach to zero with h .

It follows, therefore, that when, besides the necessary conditions [26], the moving normal approaches the fixed one in such a direction that the following conditions also have place at the same time, namely, the conditions

$$\left. \begin{aligned} \frac{d^2 A}{dx^2} + 2 \frac{d^2 A}{dx dy} \frac{dy}{dx} + \frac{d^2 A}{dy^2} \frac{dy^2}{dx^2} &= 0 \\ \frac{d^2 B}{dx^2} + 2 \frac{d^2 B}{dx dy} \frac{dy}{dx} + \frac{d^2 B}{dy^2} \frac{dy^2}{dx^2} &= 0 \end{aligned} \right\} \dots\dots [27]$$

this normal, when near to coincidence with that through the umbilicus, passes closer to the centre of the osculating sphere than it would do if the coefficients [27] were finite. In other words, this normal is more nearly a normal to the sphere than any other in the vicinity of the umbilicus.

(55.) We infer, therefore, that when the direction $\frac{dy}{dx}$ of any line issuing from an umbilicus is such as to fulfil the conditions [27] (those marked [26] are fulfilled for *every* direction), the surface in the direction of that line will lie more closely to the osculating sphere, in the immediate vicinity of the umbilicus, than in any other direction not satisfying the same equations.

If besides these equations the additional conditions arising from equating to zero the coefficients of h^2 , when these attain the limit, also have place for any particular

values of $\frac{dy}{dx}$, then such values will refer to directions in which the surface will lie still more closely to the sphere, as is evident from the foregoing reasoning.

(56.) It is of importance to observe that the conditions [27] arise from differentiating the preceding conditions [26] with respect to x and y considered as dependent, regarding $\frac{dy}{dx}$ as constant; and as the conditions [26] are equivalent to the single condition [22] at page 60, it will be sufficient to differentiate this latter under the restrictions just mentioned, in order to obtain a new condition equivalent to the conditions [27]. And as this new condition will appear under the form of an equation of the third degree in $\frac{dy}{dx}$, there will in general be at least *one* direction of proceeding from the umbilical point, in which the surface will approach with more than ordinary closeness to the osculating sphere; and, it would seem, there *may* be three directions of more than ordinary intimacy of contact. If, indeed, this equation of the third degree should, like that of the second from which it is deduced, be identical for the coordinates of the umbilical point, it is obvious from the preceding investigation, that to ascertain the particular directions of closest contact we must proceed to another differentiation, and so on till we arrive at a determinate equation, the real roots of which will make known the number and directions of the lines most nearly in contact with the osculating sphere.

The equations of the lines of curvature themselves will be obtained by integrating the differential equation involving the real component factors of this determinate equation *before* substituting for the general coordinates those of the umbilical point, and then introducing these latter for the purpose of fixing the different values of the

constant, these values being the same in number as the component factors referred to.

(57.) When, however, the umbilicus is such that, proceed in whatever direction from it we may, the lines are equally close to the osculating sphere or to the tangent plane at the point, then these successive differentiations, since they can lead to no particular values for $\frac{dy}{dx}$, must either at length exhaust the coefficients differentiated, and thus conduct to no determinate equation, or else they will furnish an equation whose roots are all imaginary. It is obvious that one or other of these circumstances must always have place at the vertex of a surface of revolution; since from this vertex we at once perceive that the lines of curvature must proceed in all directions, and with equal closeness to the osculating sphere. But in such a case as this we can never be led to roots that are all imaginary, because whatever values of $\frac{dy}{dx}$ are concealed in the differential equation [22], under the form $\frac{0}{0}$, no imaginary value can be implied; since, as already shown (53), the roots are necessarily real. At a vertex of revolution, therefore, the differentiations referred to must always eventually exhaust the coefficients differentiated; and this must always happen at a point where the curvature is perfectly uniform in all directions.

It is plain that this uniformity cannot exist all round the point except at a vertex of revolution; since, when it does exist, each normal section may be regarded as merely the successive positions of the same section in turning round the normal; and it obviously matters not to how small a distance beyond the umbilicus the uniformity may have been proved to extend. An interruption in this uniformity is the only thing that can distinguish an ordinary umbilicus from a vertex of revolution.

In the ordinary umbilicus, as the foregoing investigation proves, there will be one direction at least in which the approaching normal, immediately before coincidence with the fixed one, will be more nearly identical with this latter than in any other direction; but if, as in the case here supposed, the analytical conditions show that there is no distinction of this kind among the approaching normals, then, immediately before coincidence, the fixed normal must be symmetrically placed with respect to all the surrounding ones; so that they must all meet it in the same common point, which point ultimately becomes the centre of curvature. It is clear that a point so circumstanced must be a vertex of revolution.

(58.) In order to determine whether a surface admits of umbilical points, we must deduce from the equation of that surface the general values of p, q, r, s, t , and substitute them in the double equation

$$\frac{1+p^2}{r} = \frac{pq}{s} = \frac{1+q^2}{t} \dots\dots\dots [28];$$

which is, in reality, equivalent to two distinct equations. These, combined with the equation of the surface, will furnish three equations, which, solved for x, y, z , will make known the coordinates of the umbilical points whenever such points exist; or, by presenting these coordinates under impossible values, will show that the surface has no umbilici.

But if it should happen that the two equations [28] involve only one distinct condition, each equation being virtually no more than a repetition of the other, then the combination with the equation of the surface will be, not a point, but a *locus*; namely, a curve traced upon the surface, every point of which curve will be an umbilicus: this

curve, at every point of which a sphere osculates the surface, is called a *line of spherical curvature*.

Lastly, if no condition at all be implied in the equations [28], each side being virtually nothing but a repetition of the other, then *every* point on the surface will be an umbilicus; and since, in consequence of the identities here supposed, whatever point be taken, the subsequent differential coefficients can never become determinate for this point, it follows that lines of curvature proceed from it, not only in every direction, but with equal closeness of contact. Each point, therefore, must be a vertex of revolution; and hence the surface itself must be a sphere, so that the elaborate processes of MONGE and POISSON, to prove that the surface, every point of which is an umbilicus, fulfilling the identical conditions [28], must be a sphere, would appear to be unnecessary. Uniform closeness of contact about a point, with the osculating sphere, or with the tangent plane, marks a vertex of revolution; and if such uniformity be proved to exist about *two* points only, provided these be not diametrically opposite, it is sufficient to warrant the inference that the surface is a sphere.

(59.) The equation [22] is generally referred to as the differential equation of the lines of curvature, or rather of the projections of those lines upon the plane of x, y . To determine the lines themselves for any given surface, the expressions for p, q, r, s, t , in terms of x and y , must be deduced from the equation of that surface, substituted in [22], and then the integration of the resulting differential equation effected. The integral equation thus obtained will represent indifferently either of the lines of curvature belonging to the individual surface under examination; and in order that any one line in particular, or rather one pair of lines, may alone be represented by this integral equation, the arbitrary constants, introduced into it by integration,

must be so determined as to fix a particular point in the line sought.

But if the point be fixed first, before integration, by giving particular values to the x , y involved in the coefficients of $\frac{dy^2}{dx^2}$ and $\frac{dy}{dx}$, and in the absolute term of [22], that equation will no longer be the *differential* equation of a line of curvature through (x, y, z) ; it will be simply a common quadratic equation, the unknown term in which is represented by $\frac{dy}{dx}$, which expresses the tangent of the inclination to the axis of x of the projected directions, according to which consecutive normals intersect that at the given point. Whether or not *lines of curvature* proceed in these directions; that is, whether or not a continuous series of points proceeds from that proposed in the initial directions marked out by the values of $\frac{dy}{dx}$, taking different routes, is a question that cannot possibly be determined at this stage of the enquiry.

(60.) It is, we suspect, from not attending to this distinction—the distinction, namely, between the general equation [22], when, by applying it to particular points, it becomes a common *algebraic* equation of the second degree, and when, by leaving the points arbitrary, it is a *differential* equation of the first order and second degree,—that the principal perplexities and conflicting doctrines about the lines of curvature issuing from umbilici have arisen. Thus, the common mode of expression in reference to this subject is, that when the point under consideration is an “*ombilic*,” the equation of the lines of curvature relatively to this point will take the form $0 = 0$; which form shows, it is said, that from an ombilic there issue *an infinity of lines of curvature*, the direction of the first element of each of them being arbitrary. This is the language of LEROY, the latest and most perspicuous writer on

this subject.* POISSON, too, in his *Mémoire*, [*Journal de l'Ecole Polytechnique*, tome xiii.] entertains a similar view.† But nothing can be legitimately inferred merely from the roots of the quadratic equation [22], whether its coefficients be fixed, or take the indeterminate form $\frac{0}{0}$, beyond the simple fact of the number of values of the inclination $\frac{dy}{dx}$, not of a *curve* to the axis of x , but of the *direction* in which the consecutive normal intersects the fixed one. What, in the language of LEROY, is the first *element* of the line of curvature may, for aught the equation referred to can tell us, constitute the entire line; the so called line may be, in fact, but a point.

All that can be safely inferred from what has preceded, as respects umbilical points, is that at such points consecutive normals intersect in every direction; and that, in *one* of these directions at least, the surface is in closer contact with the osculating sphere than in any other direction.

If we were to use the language of the infinitesimal calculus, we should say that, as in this particular direction the normals intersect for *two* successive elements, if any line of curvature at all proceeds from the umbilical point, it must be in that unique direction. And as the general integral of [22] shows that two lines of curvature always pass through every other point of the surface, the law of continuity would authorize us in pronouncing that, at an umbilical point, the particular direction referred to must be the path of *one* line of curvature; and which must be

* *Analyse appliquée à la Géométrie*, p. 296. 1835.

† "Lorsque le point M sera ombilic, l'équation [14] sera identique; quelle que soit la direction déterminée par le rapport de dy à dx , la normale MN au point M sera donc rencontrée par la normale consécutive, et les droites menées par ce point dans le plan tangent seront toutes tangentes à des lignes de courbure. Mais parmi toutes ces lignes en nombre infini, il y en aura une ou plusieurs que se distingueront des autres par une propriété particulière."—*Journal de l'Ecole Polytechnique*, tome xiii., p. 218.

that into which two distinct lines, through a neighbouring point, have merged when the point has coincided with the umbilicus.

The complete integral of the differential equation [22], for any proposed surface, will, we know, from the principles of the integral calculus, be an algebraical equation, involving an arbitrary constant C in its *second power*; so that, solving this quadratic in C , for any particular values of x, y , we shall thus have *two* constants, and therefore two distinct equations, representing the curves passing through the point whose coordinates have been used to determine C . If it be possible to choose these coordinates so that C may have *equal* values, or be zero, then it is plain that the two equations will become confounded; and we shall be presented with the peculiarity just stated. If the coordinates can be such as to render C indeterminate, or of the form $\frac{0}{0}$, then the lines will be innumerable; and it is obvious that, on every other supposition, the quadratic in C must give *two* distinct values. Hence, through an isolated umbilicus, there must pass either but *one*, or else an *infinite number* of lines of curvature.

But it must not be inferred, as is customary with writers on this subject, that the merging of two lines of curvature into one implies a coincidence of the two roots of the quadratic equation [22] or [24] in the extreme case supposed, and in which the coefficients are replaced by the symbol $\frac{0}{0}$: the form of the equation forbids the inference.

The single value of $\frac{dy}{dx}$, belonging to the line of curvature at the umbilicus, is the real root of the cubic equation

$$A\left(\frac{dy}{dx}\right)^3 + B\left(\frac{dy}{dx}\right)^2 + C\left(\frac{dy}{dx}\right) + D = 0 \dots [29],$$

obtained by differentiation, as already directed (56), from

the quadratic [22], and which real root is the only one fulfilling the *two* conditions [22] and [29] for the proposed coordinates; although there is in [22] or [24] a companion root to the value thus determined of $\frac{dy}{dx}$, and which, if [24] be employed, is minus the reciprocal of that value.

It is this accompanying value of $\frac{dy}{dx}$ which here, as at every other point of the surface, marks one of the two directions of principal curvature, the other direction being that of the unique line of curvature through the umbilicus.

(61.) In order to remove from the mind of the student all suspicion, as to whether at an isolated umbilicus the roots of the cubic equation [29] can in any case be all real, it will be sufficient that he reflect, that this equation involves only those values of $\frac{dy}{dx}$ which enter into the equation [22], when the coordinates of the umbilicus are substituted in both; not indeed *all* the innumerable values of $\frac{dy}{dx}$ which in the case before us [22] furnishes, but those only which obey the general law implied in [22]; in other words, those which belong to lines of curvature; and we have just seen that the passage of *three* lines of curvature through the point is impossible. We are justified therefore in affirming, that at an isolated umbilicus two roots of the determinate equation [29] are always imaginary.

But when the umbilicus is not isolated, but is merely one of the continuous series which make up a *line of spherical curvature*, then equation [29] holds for every point in that line. It is therefore the differential equation of it; and, being of the third degree, furnishes *three* integrals corresponding to every particular point in it. It is this differential equation of the third degree that then supplies *entirely*, and not for a particular point merely, the place of the equation of the second degree which suffices for lines

of curvature in general; and just as in the case of these ordinary lines of curvature we infer that two pass through every point, so here we infer that through every point on a line of spherical curvature *three* lines in general pass, one being of course the line of spherical curvature itself.

If in the general integral of the equation [22] the constant C be determined, for one of the points on the line of spherical curvature, the two values of C will obviously be in general unequal; for to render these values equal, the irrational part of the root of the quadratic that determines C must be made zero; by which condition *particular values* only of the coordinates will be determined; and even these may be forbidden by the equation of the line of spherical curvatures, which we have seen is given by the combination of [28] with the equation of the surface. Hence the two values of C will, in general, be unequal. These therefore will imply two lines of curvature at right angles to one another, two lines uniting with the general series of lines of curvature which cover the surface. Consequently through every point on a line of spherical curvature there passes in general besides that particular line two others, of the ordinary series, at right angles to one another.

The reasoning by which DUPIN (*Développ.*, pp. 165-6,) arrives at a similar conclusion we must confess ourselves unable to comprehend.

(62.) It is contrary to the spirit of analysis to conclude from the peculiarities of an umbilicus that the greatest and least curvatures at such a point, instead of being at right angles, become confounded in direction. Whatever direction be taken at an umbilicus for that of one of these curvatures, the direction perpendicular to it will alone in strictness be that of the other. The constant and uniform law that governs these directions is, that from whatever point they proceed they preserve their perpendicularity to

one another; and to admit that when the point reaches an umbilicus this law is abruptly annulled, and that the angle changes at once from 90° to zero, is to admit so palpable a violation of the principle of continuity, a principle which essentially connects together the individual cases of every general result of analysis, that it is remarkable such an admission should have ever been made. The direction $\frac{dy}{dx}$ at an umbilicus is undoubtedly expressed under the form $\frac{0}{0}$, which, as we have seen, indicates innumerable values; these values however necessarily occur in pairs, obeying the general law referred to above; so that instead of saying with DUPIN,* that the two principal sections there become confounded, the proper inference is that these sections are innumerable. One of these, however, and only one, unites with the continuous series of values which $\frac{dy}{dx}$ passes through during the progress of a point (x, y, z) along the surface up to the umbilicus. This unique direction is, as we have seen, peculiarly a direction of principal curvature; that perpendicular to it is therefore the direction of the other curvature.

(63.) It is probable that the mistake just adverted to may owe its prevalence to the following circumstances, which appear to confirm the doctrine: 1st, it is admitted that the greatest and least curvatures at any point have the same directions as the lines of curvature issuing from that point; and 2dly, it is also admitted that at an umbilicus the lines of curvature, usually two, merge into one. At a hasty glance it does indeed seem that, if the two lines of curvature have always the same directions as the sections of maximum and minimum curvatures, when the former

* " Dans ce cas la plus grande et la moindre courbure, au lieu d'être constamment à angle droit, se confondent dans leur direction."

Développements, p. 166.

lines coalesce these directions must coalesce too. But the lines of curvature coalesce only in the sense in which an ellipse, by the continual shortening of its minor diameter, collapses into a straight line, the major diameter, and becomes confounded with it; whereas the *directions* of the fixed diameter, and the curve, at the vertex of it, never become confounded; but always preserve their mutual perpendicularity.

An examination of the lines of curvature of the ellipsoid, an examination usually presented by analysts to illustrate and confirm the ordinary doctrines on this subject, will fully support what has been now advanced. But previously to entering upon this it may be instructive to offer a few reflections on the more general theory, suggested by the foregoing investigation.

(64.) It will of course be understood by the reader, that throughout that investigation it is supposed that the several differential coefficients r , s , t , derived from the equation of the surface, do not *themselves* become susceptible of multiple values, for the particular point under consideration. This, however, they may do when for that point they take the form $\frac{0}{0}$; and, in the *Mémoire* already referred to, POISSON has entered at length into the analytical investigation of this peculiarity. We may remark, however, that it is obvious without any analytical reasoning, that if there exist a point upon a surface such that the tangent plane at that point touches the surface along n , straight lines passing through it, a normal section exists between the angle formed by every pair of these lines, of which the radius of curvature is a maximum or a minimum. For as the curvatures of the sections through the sides of the angle are each 0, and as the surface between these sides is continuous, it follows that within these limits there is a continuous series of normal sections, of

which the extremes have each the curvature 0; and consequently, in proceeding from one extreme to the other, the curvatures must increase from 0 up to a certain limit, and then must decrease down to 0; so that there must necessarily be a maximum state of curvature between every two successive lines of contact.

But it must be borne in mind that the tangent plane, in which these lines of contact are situated, is regarded as the only one passing through the point in which the lines meet, as in the ordinary theory of curvature; where the tangent plane through the point under examination is always considered to be unique, those singular points which admit of multiple tangent planes, and therefore multiple values of p, q , being excluded from the general theory. This restriction requires, in the case before us, that the tangent plane *cut* the surface in the several straight lines; dividing it into compartments about the point alternately convex and concave; just as every tangent plane divides the surface of the single-sheet hyperboloid.

It is easy to see that the portions of surface intercepted between every pair of opposite vertical angles will have curvature of the *same* kind when n is even, as in the hyperboloid referred to; and curvature of opposite kinds when n is odd. In this latter case the sections of principal curvature will each have two radii, since the normal at the point will be cut by a consecutive normal on one or the other side of the surface, according as this consecutive normal approaches the fixed one from the concave or from the convex region. The number of lines of curvature, or more accurately the number of directions in which consecutive normals intersect that at the point, may therefore be said to be n or $2n$, according as n is even or odd. With respect to the two radii of curvature of each of the sections at the point under consideration, it will be seen that they are infinite, because the section has an inflexion at the proposed point. At every such point in a plane

curve the radius of curvature may always with propriety be said to be twofold, because of the twofold character of $\frac{d^2y}{dx^2}$ at the point of inflexion. On one side of the fixed normal the algebraic sign of $\frac{d^2y}{dx^2}$ is *plus*, and on the opposite side it is minus : of these signs *that* is of course preserved at the limit which accompanies the moving normal up to that limit. As two normals or radii with opposite signs thus terminate in the normal at the point, it follows that the proper expression for this latter is $\pm \infty$; the opposite signs denoting not any particular positions in reference to the axes, but mere opposition of direction.

(65.) It has already been noticed that in surfaces presenting points of this kind, one or more of the differential coefficients r, s, t , derived from the surface, must for every such point take the form $\frac{0}{0}$; although the mere occurrence of such a form will not necessarily imply that the point under consideration has the character above described. To determine this we must ascertain by the ordinary methods what are the particular values, concealed under $\frac{0}{0}$, which unite continuously with the preceding values of the general function whence $\frac{0}{0}$ has arisen : there may possibly be but one such value.

We may lay it down as a general analytical principle that when we cannot make reference to preceding states of the function which presents itself in the form $\frac{0}{0}$, and yet know that innumerable values are inadmissible, our mode of proceeding, in the singular instance before us, is inapplicable ; or that, as the French analysts express it, " la méthode ordinaire est en défaut."

LINES OF CURVATURE OF THE ELLIPSOID.

(66.) THE equation of the ellipsoid, when referred to its principal diameters as axes of coordinates, is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \dots [30].$$

Differentiating this with respect to x and y , we have

$$\frac{dz}{dx} = p = -\frac{c^2x}{a^2z}$$

$$\frac{dz}{dy} = q = -\frac{c^2y}{b^2z}$$

$$\frac{d^2z}{dx^2} = r = \frac{c^4(y^2 - b^2)}{a^4b^2z^3}$$

$$\frac{d^2z}{dxdy} = s = -\frac{c^4xy}{a^2b^2z^3}$$

$$\frac{d^2z}{dy^2} = t = \frac{c^4(x^2 - a^2)}{a^2b^2z^3}.$$

And these values, substituted in the general differential equation,

$$\left\{ \frac{1+q^2}{t} - \frac{pq}{s} \right\} st \frac{dy^2}{dx^2} + \left\{ \frac{1+q^2}{t} - \frac{1+p^2}{r} \right\} rt \frac{dy}{dx} - \left\{ \frac{1+p^2}{r} - \frac{pq}{s} \right\} rs = 0,$$

furnish the particular differential equation

$$\frac{b^2-c^2}{b^2(a^2-b^2)}xy \frac{dy^2}{dx^2} + \left\{ \frac{a^2-c^2}{a^2(a^2-b^2)}x^2 - \frac{b^2-c^2}{b^2(a^2-b^2)}y^2 - 1 \right\} \frac{dy}{dx} - \frac{a^2-c^2}{a^2(a^2-b^2)}xy = 0 \dots [31];$$

which equation, we may remark, will belong to the single-sheet hyperboloid if the sign of c^2 be changed, and to the double-sheet hyperboloid if the signs of a^2 and b^2 be changed, that of c^2 remaining.

For abridgment, put

$$A = \frac{a^2(a^2-b^2)}{a^2-c^2}, \quad B = \frac{b^2(a^2-b^2)}{b^2-c^2} \dots [32],$$

and then the equation will become

$$\frac{xy}{B} \frac{dy^2}{dx^2} + \left\{ \frac{x^2}{A} - \frac{y^2}{B} - 1 \right\} \frac{dy}{dx} - \frac{xy}{A} = 0 \dots [A];$$

or finally

$$\frac{dy^2}{dx^2} + \left\{ \frac{B}{A} \frac{x}{y} - \frac{y}{x} - \frac{B}{xy} \right\} \frac{dy}{dx} - \frac{B}{A} = 0 \dots [33.]$$

To determine the integral of this in the most convenient manner, it will be best to deduce from it the differential equation of the second order, and then to combine the two, for the purpose of obtaining, through the elimination of one of the constants, a single differential equation more easily manageable than either.

Differentiating then the equation just given, we have

$$\frac{d^2y}{dx^2} \left(2xy \frac{dy}{dx} + \frac{B}{A} x^2 - y^2 - B \right) + \frac{dy^2}{dx^2} \left(\frac{dy}{dx} x + y \right) - \\ 2 \frac{dy^2}{dx^2} y + 2 \frac{B}{A} \frac{dy}{dx} x - \frac{B}{A} \left(\frac{dy}{dx} x + y \right) = 0,$$

or

$$\frac{d^2y}{dx^2} \left(2xy \frac{dy}{dx} + \frac{B}{A} x^2 - y^2 - B \right) + \\ \left(\frac{dy^2}{dx^2} + \frac{B}{A} \right) \left(x \frac{dy}{dx} - y \right) = 0 \dots [34]$$

The constants entering [33] and [34] may be considered to be $\frac{B}{A}$ and B; and if we eliminate the latter from [33] and [34], there results

$$\left(\frac{dy^2}{dx^2} + \frac{B}{A} \right) \left(xy \frac{d^2y}{dx^2} + x \frac{dy^2}{dx^2} - y \frac{dy}{dx} \right) = 0 \dots [35]$$

which would be satisfied by equating to zero either of the two factors on the left; but the first factor cannot possibly be zero, since A and B are quantities essentially positive. Hence it is the second factor that must be zero; that is,

$$xy \frac{d^2y}{dx^2} + x \frac{dy^2}{dx^2} - y \frac{dy}{dx} = 0;$$

or dividing by x^2 ,

$$\frac{y}{x} \frac{d^2y}{dx^2} + \frac{dy}{dx} \frac{x \frac{dy}{dx} - y}{x^2} = 0 \dots [36];$$

and multiplying by dx , with a view to the integration of this differential, we have

$$\frac{y}{x} d \frac{dy}{dx} + \frac{dy}{dx} \frac{x dy - y dx}{x^2} = 0;$$

that is,

$$\frac{y}{x} d \frac{dy}{dx} + \frac{dy}{dx} d \frac{y}{x} = 0,$$

which is obviously the differential of the product

$$\frac{y}{x} \cdot \frac{dy}{dx} = C,$$

C being an arbitrary constant.

Multiplying again by dx , preparatory to a second integration, we have

$$y dy = C x dx$$

of which equation the integral is

$$y^2 = C x^2 + C',$$

C' being another constant, not wholly arbitrary however, since this integral equation must verify the differential equation [33], to which it belongs. To ascertain then what relation exists between C and C' , put first $C \frac{x}{y}$ for $\frac{dy}{dx}$ in [33], and we get

$$C^2 \frac{x^2}{y^2} + \left(\frac{B}{A} \frac{x}{y} - \frac{y}{x} - \frac{B}{xy} \right) C \frac{x}{y} - \frac{B}{A} = 0$$

or

$$\left(C + \frac{B}{A} \right) C \frac{x^2}{y^2} - \frac{BC}{y^2} - \left(\frac{B}{A} + C \right) = 0,$$

or

$$\left(C + \frac{B}{A} \right) (Cx^2 - y^2) - BC = 0,$$

that is, putting now for y^2 its value $Cx^2 + C'$, as given by the integral,

$$(C + \frac{B}{A}) C' + BC = 0,$$

$$\therefore C' = -\frac{ABC}{AC + B}.$$

Consequently the correct integral is

$$y^2 = Cx^2 - \frac{ABC}{AC + B} \dots\dots [37]$$

in which C is the only arbitrary constant.

We thus infer that the projections of the lines of curvature upon either of the principal diametral planes are, for central surfaces of the second order, all curves of the second degree.

(67.) We shall suppose first that the plane of these projections is that of the greatest and mean diameters of the surface; that is, we shall suppose that

$$a > b > c \dots\dots [38]$$

and shall confine our attention for the present to the ellipsoid as at first proposed.

Now, in order to determine the line of curvature passing through any proposed point, (x', y', z') , the constant C must be so determined that the general equation [37] may apply to this particular point; that is, we must have

$$y'^2 = Cx'^2 - \frac{ABC}{AC + B},$$

which furnishes the following quadratic equation in C , namely,

$$Ax'^2C^2 - (Ay'^2 - Bx'^2 + AB)C - By'^2 = 0,$$

from which we get

$$C = \frac{Ay'^2 - Bx'^2 + AB \pm \sqrt{(Ay'^2 - Bx'^2 + AB)^2 + 4ABx'^2y'^2}}{2Ax'^2}$$

Now both terms under the radical in this expression being essentially positive, it follows that the two values of C will always be real and of opposite signs. Moreover, for the positive value of C the constant $\frac{ABC}{AC + B}$ will necessarily be positive, from the conditions [32] and [38]. For the negative value the same constant will be negative, because $AC + B$ is always positive, inasmuch as this expression, taking the negative value of C above, and multiplying it by the positive quantity $2x'^2$, is the same as

$$Ay'^2 + Bx'^2 + AB - \sqrt{(Ay'^2 - Bx'^2 + AB)^2 + 4ABx'^2y'^2}$$

or as

$$Ay'^2 + Bx'^2 + AB - \sqrt{(Ay'^2 + Bx'^2 + AB)^2 - 4AB^2x'^2}$$

which is obviously positive.

(68.) It is necessary to show that the two values of C in [37] give two equations of different forms, before we can conclude that the two curves passing through a point (x', y') , are really distinct in kind. The method above given is from LEROY (*Analyse Appliquée*, p. 304.) But the same results may be obtained more easily without actually solving the quadratic equation in C , and thence, by means of the artifice adopted in the final expression above, inferring the sign of $AC + B$.

For the sign of the last term in that quadratic shows us

at once, that the two values of C must have *opposite signs*; and from the known composition of the coefficient of the second term, after division by that of the first term, we infer that

$$\frac{y'^2}{x'^2} + \frac{B}{x'^2} - \frac{B}{A}$$

is the sum of the positive and negative roots. Also, since $\frac{B}{A} \frac{y'^2}{x'^2}$ is the product of the same roots, we conclude that the negative root cannot be *equal*, in absolute magnitude, to $\frac{B}{A}$, for then the positive root would be equal to $\frac{y'^2}{x'^2} + \frac{B}{x'^2}$ and thus their product would surpass its proper value; nor can the negative root *exceed* $\frac{B}{A}$ in absolute value; since, in that case, the positive root would be greater than on the former supposition, and consequently the product of the two would differ still more largely from the true product. Hence, the negative root must be *less* in absolute value than $\frac{B}{A}$, or A times this root less than B ; so that, whether the positive or the negative value of C be taken, $AC + B$ will always be positive.

(69.) We infer therefore that, in general, through every point (x', y', z') , on the surface of the ellipsoid, there pass *two* lines of curvature, whose projections on the plane of the greatest and mean diameters are represented by the equations,

$$y^2 = -Cx^2 + C'$$

$$y^2 = +C_1x^2 - C'_1.$$

The former of these equations is that of an *ellipse*; the

latter that of an *hyperbola* concentric with it, the common centre being that of the principal elliptic section of the surface on the plane of xy .

(70.) If, however, the point (x', y', z') be such as to render the irrational part of the expression for C zero; that is, if it satisfy the conditions

$$Ay'^2 - Bx'^2 + AB = 0,$$

and

$$x' y' = 0,$$

these two curves must merge into one, since the two values of C will then be identical. If such a point exist on the surface, it must have either $x' = 0$, or $y' = 0$. It cannot have the former, since for $x' = 0$ the first condition renders y' imaginary; but for $y' = 0$, that condition gives $x' = \pm \sqrt{A}$. Hence there are four of these points on the ellipsoid, the coordinates of them being

$$+ \sqrt{A}, 0, z'; - \sqrt{A}, 0, z'; + \sqrt{A}, 0, -z'; - \sqrt{A}, 0, -z',$$

in which

$$\sqrt{A} = a \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}.$$

These are the four umbilical points of the surface: they would have been as readily made known by equating to zero the coefficients and absolute term in $[A]$, the equation then giving an infinitude of values for $\frac{dy}{dx}$, only *one* of which values however points out the direction of a line of curvature. The line itself having for its projection the equation $y = 0$, the $\frac{dy}{dx}$ belonging to it is everywhere $\frac{dy}{dx} = 0$.

It thus appears that the conditions which render the values of $\frac{dy}{dx}$, at the point determined by them, innumerable, are the very same as those which cause the two lines of curvature to coalesce and form but one. It is therefore incorrect to say, with MONGE and POISSON, that the lines of curvature through such a point are innumerable, and that the particular line here adverted to differs from all the others only by a certain peculiarity. And it is equally incorrect to say with DUPIN, that the value of $\frac{dy}{dx}$ at this point is unique, and determined by the direction of the single line of curvature passing through that point; since, as already shown, there is no limit to the number of admissible values for $\frac{dy}{dx}$, when the conditions, here supposed to exist, have place.

The manner in which the single line of curvature passing through each umbilicus has been determined, has nothing in it that is peculiar, or that amounts to any departure from the method of obtaining the lines of curvature in general, passing through any other point on the surface. In all cases the coordinates of the proposed point are employed, like as we have employed those of the umbilicus, to determine the constants, and thus fix the equations of the two lines through that point; and the occurrence of equal values for these constants is all that is necessary to apprise us that the point to which they refer is an umbilicus; so that the statement of DUPIN, that the ordinary method of determining the lines of curvature through such points *fails*,—unless, indeed, they be upon a line of spherical curvature,—appears to have been hastily made.*

(71.) We have just seen that there are four points on the ellipsoid, in reference to which the two values of the con-

* See *Développements de Géométrie*, p. 359.

stant C , in the integral equation of the pair of lines passing through each, become identical, these values being each zero. From the mere form of the quadratic equation which determines C , or indeed from that of its final term only, we might have predicted that if such identical values were possible, they must necessarily be zero; for as that final term is essentially negative after dividing by the coefficient of C^2 , it is impossible that *equal* roots can exist except in the extreme case of $C = 0$; and even then the roots, strictly speaking, are $C = 0$ and $C = -0$.

(72.) The evanescence of the irrational part of the expression which exhibits the two roots of a quadratic equation is the ordinary indication of the equality of those roots. But in extreme cases—those, namely, in which the entire numerator vanishes with the irrational part of it—we must not hastily conclude merely that two roots have merged into one; for it is necessary to such conclusion that the denominator be not, like the numerator, evanescent. Our inference, as to the two values of C being zero, when the numerator is so, is valid, because the denominator remains finite. But if we take the equation [A], namely,

$$\frac{xy}{B} \frac{dy^2}{dx^2} + \left\{ \frac{x^2}{A} - \frac{y^2}{B} - 1 \right\} \frac{dy}{dx} - \frac{xy}{A} = 0,$$

and equate to zero the irrational part of the expression for $\frac{dy}{dx}$; or which is the same thing if we take the ordinary criterion of equal roots, namely,

$$\left\{ \frac{x^2}{A} - \frac{y^2}{B} - 1 \right\}^2 + \frac{4x^2y^2}{AB} = 0;$$

and infer that when this is satisfied, the two values of $\frac{dy}{dx}$,

which in the absence of this condition are distinct, will then become confounded, we shall commit error. For this condition cannot hold unless we have the two conditions

$$\frac{x^2}{A} - \frac{y^2}{B} - 1 = 0$$

$$xy = 0$$

inasmuch as the sum of two squares cannot be zero except each be zero separately. Now these two conditions cannot have place, for real values of x and y , unless $y = 0$; and since the denominator in the expression for the root, and which denominator has been disregarded, is $2 \frac{xy}{B}$, the true inference is, not that in the supposed circumstances the two roots have become one, but that these roots have become innumerable; the expression for them being

$$\frac{dy}{dx} = \frac{\frac{x^2}{A} + \frac{y^2}{B} - 1}{2 \frac{xy}{B}} = \frac{0}{0}$$

The two conditions above, being identical with those at (70), determine the umbilical points, as this last indeterminate form for $\frac{dy}{dx}$ would imply. These same conditions however have been erroneously supposed to indicate *equal*, instead of *innumerable* values for $\frac{dy}{dx}$. The umbilical points have thus been correctly determined, although not as a consequence of the true interpretation of $\frac{dy}{dx}$ in reference to them.

The mistake here adverted to was committed by MONGE, the illustrious originator of these speculations; and although the error was pointed out by DUPIN, (*Développe-*

ments, p. 185,) who does not however supply the correct conclusion, it has been repeated by later writers. (See *Transactions of the Royal Irish Academy*, Vol. XIV. p. 89.)*

(73.) The line of curvature through the four umbilici, having for its equation $y = 0$, is the principal ellipse in the plane of the greatest and least axes. If we conceive a point on this ellipse to move towards an umbilicus, it will thus describe one of its own lines of curvature; the second, perpendicular to this, must, by the law of continuity, gradually collapse as the moving point approaches the umbilicus, the tangent $\frac{dy}{dx}$, of the collapsing curve at its vertex, still remaining ∞ . When the umbilicus is actually reached, the curve closes, and becomes confounded with the line of curvature described by the point; the tangent at the vertex of this closed curve is still however perpendicular to the axis into which it has merged. But these particulars will perhaps become more intelligible from an inspection of the lines of curvature themselves.

To project these on the plane of xy , it will be convenient to put the general equation [37] of the projections under the form

* Dupin remarks, in reference to the foregoing expression for $\frac{dy}{dx}$, that as it assumes the form $\frac{0}{0}$ nothing can be inferred from it; and, like as in other parts of his work, affixes to this symbol only the particular value subsequently determined from it by the application of the calculus. But this view of the form $\frac{0}{0}$ is an erroneous one. If no more than a particular value were concealed under that form the fraction would have a vanishing factor common to numerator and denominator, which however is not the case. Numerator and denominator vanish in consequence of two distinct hypotheses or conditions; and these are such as to render the original equation indeterminate, and therefore the values of $\frac{dy}{dx}$ innumerable. (See the *Differential Calculus*, p. 71.)

$$\frac{x^2}{m^2} \pm \frac{y^2}{n^2} = 1 \dots\dots [39]$$

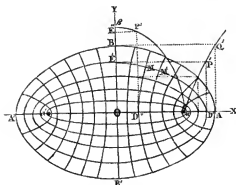
by writing for the essentially negative quantity, $\frac{AB}{AC+B}$, the symbol $-m^2$; and for $\frac{ABC}{AC+B}$, which, as we have seen, is positive for one line of each pair, and negative for the other, the symbol $\pm n^2$; the quantities m and n will then be the semiaxes of the two curves. These quantities are also otherwise connected with each other by a remarkable relation—the relation between the quantities for which we have just substituted them; and which is seen at a glance to be expressed by the equation

$$\frac{m^2}{A} \mp \frac{n^2}{B} = 1 \dots\dots [40]$$

the upper sign having place, as in the preceding equation, for the elliptic lines of curvature, and the lower for the hyperbolic.

From this it appears that the semidiameters m, n , of any one of the projections [39], are respectively equal to the coordinates m, n of some point in the corresponding curve [40]; so that taking the upper sign in these equations, the coordinates of a point (m, n) in the hyperbola [40] will exhibit the length of the semidiameters m, n of one of the elliptic projections [39]; and taking the lower sign, the coordinates of a point (m, n) in the ellipse [40] will be the lengths of the semidiameters of one of the hyperbolic projections [39].

Hence both series of projections may be constructed, by help of the *auxiliary* hyperbola and ellipse, [40], as follows:



On the semiaxes OA, OB of the principal ellipse, coinciding with the plane of projection, set off the distances

$$O\alpha = \sqrt{A} = a \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}$$

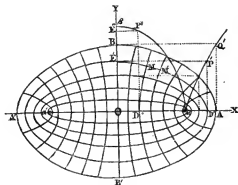
$$O\beta = \sqrt{B} = b \sqrt{\frac{a^2 - b^2}{b^2 - c^2}}$$

and with these as semidiameters construct the auxiliary hyperbola $aP'Q'$ whose real vertices shall be upon $A'A$, and the auxiliary ellipse $aP''\beta$.

Take any point P in the hyperbola; transfer its coordinates PE', PD' to OD', OE' , and with these as semidiameters construct an ellipse $E'MD'$; this will be the projection of one of the lines of curvature. In a similar manner may every one of the elliptic projections be constructed.

(74.) For the hyperbolic projections we must transfer the coordinates, $P'E'', P'D''$ of each point P' in the auxiliary ellipse, to OD'', OE'' ; and with these as semidiameters construct an hyperbola $D''M$, whose real vertices are upon $A'A$; this will be one of the hyperbolic projections and others may be determined in like manner.

The real diameter $2Oa$ or aa' , common to the auxiliary ellipse and hyperbola, is the distance between the umbilical points, projected on the plane which receives the several lines of curvature (70); and it is plainly this line aa' into which the flattening ellipses ultimately merge when P' approaches towards, and at length reaches the vertex a of the auxiliary hyperbola.



In like manner as the point P'' , moving along the auxiliary ellipse, approaches towards the same vertex, the imaginary axis OE'' of the hyperbolic projection continually diminishes to zero, and the real axis OD'' continually increases up to its limit Oa ; hence, when this is reached, each hyperbolic branch closes, and the curve then merges into the two straight lines $aA, a'A'$. The lines of curvature therefore thus circulate as it were round the umbilical points, always turning their concavities towards them. On the surface these lines are of course all curves returning into themselves, as the surface is everywhere limited; the hyperbolic arcs therefore into which one series of them is projected, are limited by this circumstance within the bounds of the principal diametral plane which receives

these projections; so that the auxiliary hyperbola is not required beyond the point Q' . The curves of double curvature, which furnish the hyperbolic projections, after passing round the umbilicus of which α is the projection, widen, embrace the vertex A of the surface, then contracting pass round the other umbilicus, having the same projection, and return into themselves.

If we take D' any point in $A\alpha$, then $D'P'$ and AA' will be the projections of the tangents, or directions, of the lines of curvature through the point which D' represents. These directions are at right angles, and they obviously continue so during the entire progress of D' up to α ; for D' continues to be the vertex of the collapsing ellipse even up to α , at which point the ellipse actually closes and becomes confounded with the line $\alpha\alpha'$.

If our moving point were taken between these limits, as at D'' ; then in like manner the directions $D''P'$, AA' , continue at right angles during the progress of D'' up to α ; at which point the hyperbola closes and becomes confounded with the lines αA , $\alpha'A'$.

Hence it may be inferred that at the umbilical point the two lines of curvature, ordinarily elliptic and hyperbolic, become, not confounded with each other, for strictly speaking they are still distinct, but, confounded with the unique line of curvature through the point; one line of curvature with one portion of this unique line, and the other with another portion; the remaining portion of the entire line is that into which the other branch of the ultimate hyperbola degenerates.

(75.) It is obvious that if an ellipsoid be cut by a plane through the mean axis BB' , and parallel to the tangent plane at an umbilicus, the section must be a circle; if this section incline towards the greatest principal section there will arise an ellipse having the mean axis of the

ellipsoid for its minor diameter; but if it incline in the other direction, towards the least principal section, the ellipse furnished will have the same mean axis for its major diameter; the ellipses thus change their character when the revolving plane, through the mean axis, passes through an umbilicus. As the tangent plane parallel to this revolving plane in any position always gives an indicatrix similar to the section made by that plane, it follows that if a point on the unique line of curvature, and between the two umbilici a, a' , move towards one of these points, the sections of *greatest* curvature will, for every position, be the principal section containing the greatest and least axes of the ellipsoid; and consequently the section perpendicular to this will always be the *least*. But when the moving point passes an umbilicus, the *least* section will, on the contrary, be that of the greatest and least axes of the surface, and that perpendicular to it will be the greatest.

On these opposite sides of the umbilicus therefore the character of the surface changes; although at any other two points on a line of curvature, and symmetrically posited relatively to the umbilicus (and taken in its neighbourhood or not), the surface will exhibit equal or symmetrical curvatures, as the lines of curvature show.

If the minor diameter of the ellipsoid—and which is the minor diameter of the ellipse on which the umbilici are situated—be conceived to lengthen till it become equal to the mean diameter, the umbilici near A will approach towards and unite in that vertex, as is plain from the expression for Oa ; in like manner the other pair of umbilici will approach towards and meet in the vertex A' : the surface will then become an ellipsoid of revolution; and the curvature about the united umbilici being uniform in all directions, lines of curvature will proceed from them in all directions.

(76.) By differentiating the general equation,*

$$Axy \frac{dy^2}{dx^2} + (Bx^2 - Ay^2 - AB) \frac{dy}{dx} - Bxy = 0 \dots [41]$$

which at the umbilicus α , where $y = 0$ and $x = \sqrt{A}$, has zero coefficients, we have

$$Ax \frac{dy^2}{dx^2} - Ay \frac{dy^2}{dx^2} + Bx \frac{dy}{dx} - By = 0,$$

and this, for the same values of y and x , is no longer indeterminate; but is decomposable into the two determinate equations

$$\frac{dy}{dx} = 0; \quad A \frac{dy^2}{dx^2} + B = 0.$$

The first of these marks the direction of the unique line of curvature; the second gives imaginary values for $\frac{dy}{dx}$.

If the surface be of revolution, about the axis of z , that is, if $A = B$, the differential equation [41], after dividing by B , remembering that in this case $A = 0$, becomes

* The student will not fail to bear in remembrance that the differentiation spoken of here, and in other places, in reference to the differential equation of lines of curvature, affects only the *coefficients* of $\frac{dy}{dx}$, and the term independent of it; $\frac{dy}{dx}$ itself being regarded as constant. These coefficients, being in the proposed hypothesis all zeros, the values of $\frac{dy}{dx}$, if expressed in functions of them, would necessarily take the form $\frac{0}{0}$; that particular value, concealed under this form, which obeys the law implied in the general expression whence $\frac{0}{0}$ has arisen is, as we know, educed by differentiating numerator and denominator of that general expression, and introducing the proposed hypothesis into the results. And it is this that is effected, virtually, by differentiating the coefficients at once, as in the text.

$$xy \frac{dy^2}{dx^3} + (x^2 - y^2) - \frac{dy}{dx} - xy = 0 \dots\dots [42],$$

which is indeterminate for $x = 0, y = 0$. Hence, differentiating the vanishing coefficients, we have

$$x \frac{dy^3}{dx^3} - y \frac{dy^2}{dx^3} + x \frac{dy}{dx} - y = 0.$$

This also for the same values of x and y is equally indeterminate: therefore, differentiating again, we get

$$0 \frac{dy^3}{dx^3} + 0 \frac{dy}{dx} = 0;$$

which, as the coefficients are exhausted, shows that the lines of curvature issuing from the point $x = 0, y = 0$, are innumerable, and equally close to the osculating sphere (57). This merely confirms what the general theory has already established for every surface of revolution.

(77.) The general differential equation [42] of the lines of curvature of such a surface, when of the second order, is integrated with great ease; for dividing by xy , it becomes

$$\frac{dy^2}{dx^3} + \left(\frac{x}{y} - \frac{y}{x}\right) \frac{dy}{dx} - 1 = 0,$$

a quadratic equation, in which the two roots are actually exhibited within the parenthesis, inasmuch as the product of these gives the absolute term -1 . Hence

$$\frac{dy}{dx} = -\frac{x}{y}, \therefore y \, dy = -x \, dx,$$

$$\frac{dy}{dx} = \frac{y}{x}, \therefore \frac{dy}{y} = \frac{dx}{x}.$$

The integral of the first of these is

$$y^2 = -x^2 + C, \therefore y^2 + x^2 = C.$$

The integral of the second is

$$\log y = \log C_1 x, \therefore y = C_1 x.$$

Consequently, the projections of the lines of curvature are, for the first series, all circles described round the origin as centre; and for the second series straight lines radiating from this common centre, and, consequently, crossing the circles at right angles. Through any proposed point (x', y', z') , only one circle and one straight line pass: the constant C_1 for this point is $C_1 = \frac{y'}{x'}$, the straight line being $y = \frac{y'}{x'} x$. When the point is the vertex of revolution, $x' = 0, y' = 0$; and the equation is $y = \frac{0}{0} x$, the lines from that point being innumerable.

(78.) If the plane on which the lines of curvature of the ellipsoid, instead of being that of the greatest and mean axes, were the plane of the least and mean axes, we might make all our preceding reasonings and deductions apply, not by changing the plane of xy , which has hitherto been employed, for that of yz ; but, preserving the plane of xy , as that of projection, by changing the relative magnitudes of the semiaxes a, b, c ; that is, by supposing

$$c > b > a,$$

under which conditions A and B will be positive as before, and the general equation [33] of the lines of curvature, thus preserving the same form, will conduct to like results as to the elliptic and hyperbolic curves into which these lines of curvature are projected.

But if the plane of projection be that containing the greatest and least axes; that is, still making the plane of projection that of xy , if the conditions be

$$a > c > b;$$

then, since B no longer continues positive, the general differential equation becomes modified, and may therefore refer to curves of a different kind. We shall proceed then to investigate the nature of the projections when the plane which receives them is that in which a and b are situated, these being considered as coincident with the axes of x and y , and as conforming to the above conditions.

Put, as in the former case,

$$A' = \frac{a^2(a^2 - b^2)}{a^2 - c^2} = A; \quad B' = \frac{b^2(a^2 - b^2)}{c^2 - b^2} = -B;$$

then, as before, the integral equation will be

$$y^2 = Cx^2 + C' \dots\dots[43],$$

the only difference between this and that formerly obtained consisting in the equation of condition which connects the constants C, C' ; which equation obviously differs from the former in having $-B'$ in place of B . Making this substitution then, we have in the case before us

$$C' = \frac{A'B'C}{A'C - B'}$$

and consequently the general integral equation of the lines of curvature is

$$y^2 = Cx^2 + \frac{A'B'C}{A'C - B'} \dots\dots[44]$$

C being the arbitrary constant determinable for any particular pair of these lines by means of the coordinates of the proposed point whence they are to issue. But wherever this proposed point be situated it may be proved that C must invariably be *negative*, and consequently that C' must always be positive. For taking any point (x', y', z') the value of C in terms of x', y' and the constants A', B' is, as at (67)

$$C = \frac{A'y'^2 + B'x'^2 - A'B' \pm \sqrt{\{ (A'y'^2 + B'x'^2 - A'B')^2 - 4A'B'x'^2y'^2 \}}}{2A'x'^2}$$

where the rational part of the numerator is obviously greater in absolute value than the irrational quantity connected with it. Now this rational part is negative for every pair of values of x', y' ; that is to say, we must always have the condition

$$A'y'^2 + B'x'^2 < A'B'$$

or

$$\frac{y'^2}{B'} + \frac{x'^2}{A'} < 1.$$

For whatever point on the ellipsoid be projected on the plane of x, y , that is, the plane of a, b , the x of it can never exceed a , nor the y of it b ; and it is of course impossible that these extreme values can simultaneously exist; nevertheless, if we substitute even them for x' and y' , in this last expression, still the result will be only a unit; for

$$\frac{b^2}{B'} + \frac{a^2}{A'} = \frac{c^2 - b^2}{a^2 - b^2} + \frac{a^2 - c^2}{a^2 - b^2} = 1,$$

hence in every possible case the result is less than a unit. Consequently, as the value of C is always negative, and

that of the other constant in [43] always positive, it follows that the projections of the lines of curvature are all *ellipses*.

If we put in the general equation [44]

$$\frac{A'B'C}{A'C - B'} = +\pi^2; \frac{A'B'}{A'C - B'} = -m^2$$

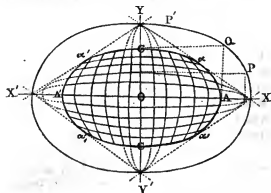
it will take the form

$$\frac{x^2}{m^2} + \frac{y^2}{n^2} = 1,$$

the semiaxes m, n of each ellipse being in virtue of the substitutions just made related to one another by the condition

$$\frac{m^2}{A'} + \frac{n^2}{B'} = 1 \dots [45],$$

so that they are always determined by the coordinates of a point (m, n) in the auxiliary ellipse [45] constructed on the semidiameters $OX = \sqrt{A'}$ and $OY = \sqrt{B'}$.



(79.) By referring to the values which A' and B' represent, we see that these semiaxes exceed those of the prin-

cipal section of the surface upon which the lines of curvature are to be projected. There is one point Q in the auxiliary ellipse of which the coordinates are equal to the semi-axes of the principal section referred to; for putting in the equation [45] a for m there is found to result b for n ; and hence this principal section is one of the lines of curvature. But as the abscissa of every point P in the auxiliary ellipse between Q and X is longer than that of Q , it follows that the ellipses determined by the coordinates of these points will extend towards X beyond the limits of the surface, or of the principal section which terminates the projections; and in like manner, since the ordinate of every point P' , between Q and Y , exceeds that of Q , the ellipses determined by such points will extend in the direction OY beyond the bounds of the surface. With the exception therefore of the principal section ACA' the lines of curvature will have for their projections only portions of ellipses. One series of these projections, determined by that part of the auxiliary ellipse which is between X and Q , will want the segments towards X and X' ; the other series crossing these at right angles, and determined by that part of the auxiliary ellipse which is between Q and Y , will want the segments towards Y and Y' . The initial ellipse in the first series, furnished by the point X in the auxiliary ellipse, is obviously the straight line XX' ; and the ultimate ellipse in the second series, furnished by the point Y in the auxiliary ellipse is the straight line YY' ; so that the three principal sections of the ellipsoid are lines of curvature.

(80.) It is of importance to notice that the general equation of these lines of curvature has been derived from the integral of the differential equation [36], which was substituted for the equation [35] because the factor $\frac{dy^2}{dx^2} + \frac{B}{A}$ could not be zero without contradicting the hy-

pothesis as to the signs of A and B, these quantities being necessarily positive. But in the present case we have no such warranty for rejecting the corresponding factor, which, because $A = A'$, and $B = -B'$, is $\frac{dy^2}{dx^2} - \frac{B'}{A'}$, furnishing the additional solution

$$\frac{dy}{dx} = \pm \sqrt{\frac{B'}{A'}}$$

This, therefore, is another integral of the differential equation of the second order [35], and by which that differential equation is evidently satisfied. It contains no arbitrary constant, nor is it implied in the general integral involving such a constant, before deduced. It is thus the *singular solution* of the equation referred to.

By substituting this value of $\frac{dy}{dx}$ in the original differential equation of the first order [33] we get the new integral equation.

$$A'y^2 + B'x^2 \pm 2xy\sqrt{A'B'} = A'B',$$

involving no arbitrary constant, and distinct from the general integral [37]. This, therefore, is the *singular solution* of the proposed differential equation.

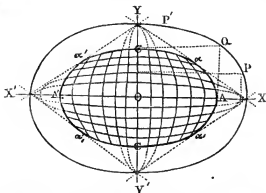
This singular solution is, we know, (see *Integ. Calculus*, p. 237,) when interpreted geometrically, the line which touches or embraces all the curves comprised in the general integral. It is in the present case decomposable into the four straight lines represented by the equations

$$y\sqrt{A'} \pm x\sqrt{B'} = +\sqrt{A'B'},$$

$$y\sqrt{A'} \pm x\sqrt{B'} = -\sqrt{A'B'},$$

which are readily seen to be the four supplemental chords

connecting the four principal vertices of the auxiliary ellipse.



These four chords therefore touch all the elliptic projections of the lines of curvature. We already know that the two lines of curvature through an umbilicus degenerate into two opposite arcs of the principal ellipse $ACA'C'$; and as the supplemental chords just determined touch all the lines of curvature, it follows that the two from each umbilicus must be touched at their junction, that is, at the umbilicus itself. Hence the points a, a_1, a', a'_1 , where the supplemental chords touch the principal ellipse are the four umbilici of the surface. Or we may infer that these points are the umbilici from considering that as the supplemental chords are the loci of the consecutive intersections of the varying ellipses, they must touch the principal ellipse at the points in which the intersections of a neighbouring ellipse with it settle, when this latter, by varying, comes to coincide with the former. These consecutive intersections are evidently the umbilici; because the two curves are distinct only so long as they cross each other at points which are not umbilici; the umbilici alone are the points at which, when the intersections merge into them, the curves become confounded, or, as it is usually expressed, consecutive.

LINES OF CURVATURE OF THE HYPERBOLOID.

(81.) THE projections of the lines of curvature of the other central surfaces of the second order may be very easily effected from the foregoing examination of the ellipsoid, as we have already intimated. Thus in the double-sheet hyperboloid we shall only have to change the signs of a^2 and b^2 , in the equation [31], and we shall thus get for the semiaxes of the auxiliary hyperbola and ellipse, the values

$$\sqrt{A} = a \sqrt{\frac{a^2 - b^2}{a^2 + c^2}}$$

$$\sqrt{B} = b \sqrt{\frac{a^2 - b^2}{b^2 + c^2}}$$

and for the equations of these curves, m , n , being as at (73), put for the x , y of any point in either, we should have

$$\frac{n^2}{B} \mp \frac{m^2}{A} = 1.$$

The real axis of the auxiliary hyperbola will in this case coincide with axis of y , or with the principal diameter $2b$ of the surface. These auxiliary curves being constructed, the elliptic and hyperbolic projections which, as in the ellipsoid, will still be analytically represented by the equations [39] may be traced as before: they will present to the eye an appearance exactly similar to those already exhibited for the ellipsoid.

The hyperboloid of a single sheet will be represented by our original equation [30], by changing the sign of c^2 ; so that the semiaxes of the auxiliary hyperbola and ellipse will be expressed as before, and the equations of the curves themselves will correspond with those already exhibited for the ellipsoid. The projections will present an appearance similar to those for the ellipsoid.

(82.) The double-sheet hyperboloid has, like the ellipsoid, four umbilical points, two on each sheet, round which the projections of the lines of curvature circulate, presenting always their concavities towards those points.* And although, as just remarked, the projections in the single-sheet hyperboloid present the same general appearance, yet the elliptic sections here never collapse into a straight line and become confounded with the major diameter, since these sections cannot contract beyond the *throat* of the surface, within which no projections can be situated. At the vertices of the major diameter of this limiting ellipse the hyperbolic projections close and become straight lines, having the limiting ellipse just mentioned for the second line of curvature of each.

* The four umbilici being determined exactly in the same manner as those of the ellipsoid are found at page 85, it is unnecessary to repeat the investigation.

LINE OF CURVATURE OF THE PARABOLOID.

(83.) LET the equation of the paraboloid be

$$\frac{x^2}{P'} + \frac{y^2}{P} = 2z,$$

P and P' having the same or different signs according as the paraboloid is elliptic or hyperbolic. By differentiating we have

$$\frac{dz}{dx} = p = \frac{x}{P'}, \quad \frac{dz}{dy} = q = \frac{y}{P},$$

$$\frac{d^2z}{dx^2} = r = \frac{1}{P'}, \quad \frac{d^2z}{dxdy} = s = 0,$$

$$\frac{d^2z}{dy^2} = t = \frac{1}{P}.$$

These values, substituted in the general differential equation [22], give for the lines of curvature of the paraboloid projected upon the tangent plane at its vertex, or upon any plane parallel to this, the equation

$$\frac{xy}{P^2P'} \frac{dy^2}{dx^2} + \left\{ \frac{x^2}{P^2P'} - \frac{y^2}{P^2P'} + \frac{1}{P} - \frac{1}{P'} \right\} \frac{dy}{dx} - \frac{xy}{P^2P^2} = 0;$$

or,

$$\frac{dy^2}{dx^2} + \left\{ \frac{P}{P'} \frac{x}{y} - \frac{y}{x} - \frac{P(P-P')}{xy} \right\} \frac{dy}{dx} - \frac{P}{P'} = 0.$$

This equation, by putting

$$A = P'(P - P'), \quad B = P(P - P'),$$

becomes the very same as that marked [33], the only difference being in the numerical values of A, B. It may therefore be written

$$\frac{dy^2}{dx^2} + \left\{ \frac{B}{A} \frac{x}{y} - \frac{y}{x} - \frac{B}{xy} \right\} \frac{dy}{dx} - \frac{B}{A} = 0.$$

(84.) Let the hyperboloid be elliptic; that is, let P and P' be both positive, and let $P > P'$; then, as in the ellipsoid, A and B are both necessarily positive, and consequently all the conclusions follow from this equation that were before obtained from it for the ellipsoid. We infer, therefore, that the lines of curvature of the elliptic paraboloid, when projected upon the tangent plane at the vertex, furnish, as in the ellipsoid, a series of ellipses crossed at right angles by a series of hyperbolas, the concavities of both series being turned towards the two umbilical points. The auxiliary hyperbola and ellipse which serve to project these curves have for their semiaxes the values

$$\sqrt{A} = \sqrt{P'(P - P')}$$

$$\sqrt{B} = \sqrt{P(P - P')},$$

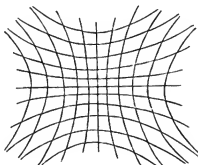
and the projections themselves are exhibited in the diagram at page 91. The preceding values of $\pm \sqrt{A}$ express, as in the ellipsoid, the distances of the origin, measured along the axis of x, from the projected umbilici.

(85.) Next, let the paraboloid be hyperbolic, then one of the coefficients in its equation must be negative: let P be

negative, then B is positive and A negative, and the integral of the preceding differential equation will differ from that marked [37] in respect to signs: it will be

$$y^2 = Cx^2 + \frac{ABC}{B-AC} \dots\dots[46];$$

and it is plain, whatever point (x', y') be taken to determine the constant C , that this constant can never be negative, inasmuch as such a negative value would render y' imaginary.



If the equation [46] be reduced to a quadratic in C , and then solved, the values of C for any proposed point, although, as just seen both positive, are nevertheless always such as to render the corresponding values of $B-AC$, the one positive and the other negative. For referring to the expression for C at page 83, and changing the sign of A to adapt it to the case before us, it is easily seen that

$$\begin{aligned} 2x'^2(AC-B) &= Ay'^2 - Bx'^2 + AB \mp \sqrt{\{(Ay'^2 + Bx'^2 + AB)^2 - 4ABx'^2y'^2\}} \\ &= Ay'^2 - Bx'^2 + AB \mp \sqrt{\{(Ay'^2 - Bx'^2 + AB)^2 + 4ABx'^2y'^2\}}, \end{aligned}$$

which is evidently positive or negative according as the irrational part is positive or negative. Hence $AC-B$ is positive for one of the values of C , and negative for the other.

It follows, therefore, that the projections of the lines of curvature upon the plane of xy , or upon any plane perpendicular to the axis, are all hyperbolas, of which the equations are

$$\frac{y^2}{n^2} - \frac{x^2}{m^2} = \pm 1;$$

the auxiliary hyperbolas by which these are constructed having for equations

$$\frac{n^2}{B} - \frac{m^2}{A} = \mp 1.$$

These denote a pair of conjugate hyperbolas, the absolute lengths of the axes of which are, seeing that P takes the sign minus,

$$\sqrt{A} = \sqrt{P'(P+P')}$$

$$\sqrt{B} = \sqrt{P(P+P')}.$$

These auxiliary hyperbolas are similar to the hyperbolic sections of the proposed surface, made by planes parallel to that of xy . For putting any constant value for z in the equation of that surface, we find that the axes of the section at that distance from the plane of xy , are to each other as $\sqrt{P'}$ to $\sqrt{P}$, and this is the ratio of the axes $2\sqrt{A}$, $2\sqrt{B}$.

NOTE.

THE expression for R , the radius of curvature of any normal section of a curve surface, deduced at page 49, is obtained on the hypothesis that the plane of xy touches the surface at the point to which R refers. By help of the equation of the projected indicatrix, given at page 56, a more general expression for R may be easily derived; that is, an expression independent of the position of the rectangular coordinate planes.

The equation referred to is

$$r(x-a)^2 + 2s(x-a)(y-b) + t(y-b)^2 = 2h.$$

where h , and the several factors within the parenthesis, are taken in the limit. This h however is not the same as the h which enters into our fundamental expression for the radius of curvature, at page 45,—that being on the normal, and this on the axis of z . The cosine of the angle γ , between the two, is (*Differential Calculus*, p. 206),

$$\cos \gamma = \frac{1}{\sqrt{1+p^2+q^2}}$$

so that the h in the foregoing equation must be multiplied by this in order to produce the h hitherto employed in the expression for R . Calling therefore that semidiameter of the indicatrix, through which the normal section passes, k , we have by the usual formula (page 45),

$$\begin{aligned} R &= \frac{k^2}{\frac{r(x-a)^2 + 2s(x-a)(y-b) + t(y-b)^2}{\sqrt{1+p^2+q^2}}} \\ &= \frac{k^2 \sqrt{1+p^2+q^2}}{r(x-a)^2 + 2s(x-a)(y-b) + t(y-b)^2} \end{aligned}$$

Now it is obvious that $(x - a)$ and $(y - b)$ are the projections on the axes of x and y of each semidiameter k ; so that putting α and β for the angles at which k is inclined to the axes of x and y , respectively, the preceding expression for R may be changed into

$$R = \frac{k^3 \sqrt{(1 + p^2 + q^2)}}{rk^2 \cos^3 \alpha + 2sk^2 \cos \alpha \cos \beta + tk^2 \cos^3 \beta}$$

$$= \frac{\sqrt{(1 + p^2 + q^2)}}{r \cos^3 \alpha + 2s \cos \alpha \cos \beta + t \cos^3 \beta}$$

Just as in the case of plane curves, so here, the radical which enters into the expression for R has been the source of some perplexity and confusion in the doctrine of surfaces. Instead of viewing this radical in its proper light, as explained in the first Dissertation, and rejecting the double sign of it as a superfluous appendage, introduced into the analysis by giving a wider interpretation to our symbols than the restrictions of the problem warrant, analysts have sought to connect this double sign with the direction of curvature, and have recommended one or other of the two to be preserved according to circumstances.*

LEROY† states that if we agree to take this radical always *positively*, the formula will indicate at once both the amount and the direction of the curvature of each normal section. But this convention is opposed to the pre-established theory of the sign in plane curves, where the general expression is preceded by a *minus*; and it is a plane curve in the case before us. The proper sign prefixed to R should therefore be *minus*, and not *plus*.

* See Dupin. *Développemens*, p. 144.

† *Analyse Appliquée à la Géométrie*, p. 263.

This sign however has no reference to the radical; but to the hypothetical conditions implied in the general investigation of R , as already explained. In our reasonings in the present dissertation we have taken account only of the absolute value of R , for the purpose of deducing general properties of the surface. At the outset (Art. 31) the algebraic sign of R was suppressed; and its absolute length only made to enter our investigations. If this sign be restored the expression [7] at page 45 will be $R = -\frac{a^2}{2h}$ and, consequently, that deduced above is

$$R = -\frac{\sqrt{(1+p^2+q^2)}}{r \cos^2 \alpha + 2s \cos \alpha \cos \beta + t \cos^2 \beta}$$

If the plane of xy were a tangent at the point to which R refers, and moreover, if the normal section passed through the axis of x , then it is plain that the numerator of this expression for R would be reduced to 1; and the denominator to r , since in the case supposed $\cos \alpha = 1$, and $\cos \beta = 0$; so that we should have

$$R = -\frac{1}{r} = -\frac{1}{\frac{d^2z}{dx^2}}$$

as at page 28. If, as LEROY proposes, the sign were to be taken *plus*, there would then be a discrepancy between two expressions which ought to be identical.

We take this opportunity of remarking, in reference to the statement at p. 73, that when the two quantities under the radical are not, as at p. 83, both essentially positive, they will, when equated to zero, represent a *curve*; and the *particular values*, adverted to in the text, will then belong to the intersections of the curve, of which this is the projection, with the line of spherical curvature.

DISSERTATION III.

ON THE INCOMMENSURABILITY OF THE CIRCUMFERENCE OF A CIRCLE TO ITS DIAMETER; AND ON THE SUMMATION OF CERTAIN CLASSES OF INFINITE SERIES DEPENDENT ON THE NUMERICAL VALUE OF THE SEMICIRCLE.

DISSERTATION III.

ON THE INCOMMENSURABILITY OF THE CIRCUMFERENCE OF A CIRCLE ; AND ON THE SUMMATION OF CERTAIN CLASSES OF INFINITE SERIES.

(86.) MANY attempts have been made by modern analysts to demonstrate the incommensurability of the number which expresses the ratio of the circumference of a circle to its diameter ; and thus to put an end to all hope of effecting the numerical rectification of the circumference, or the numerical quadrature of the inclosed circle, in finite terms.

Of these attempts we believe but two, at most, have been regarded as completely successful. Of these, one was published by LAMBERT, in the *Memoirs of Berlin* for 1761, and the other by LEGENDRE, in Note iv. at the end of his *Geometry*. The demonstration of LAMBERT has not met with very general acceptance among mathematicians ; and since the publication of LEGENDRE's method of proof, which has entirely superseded the former, it is but very seldom referred to by the more recent writers on the circle.

The demonstration of LEGENDRE, though no doubt satisfactory, is nevertheless long and intricate ; resting upon several subordinate analytical theories which greatly retard and complicate the proof ; so that a short and easy demonstration, dependent solely upon the well-known

fundamental principles of analysis, is still a desideratum in the complete discussion of a problem which has occupied the special attention of mankind ever since the origin of geometry.

Whether the attempt here made to supply such a demonstration fully accomplishes the object desired, it must be left with analysts to determine. On the score either of length or of intricacy, the method of proof here proposed is certainly free from objection; it is as short and as elementary as can well be expected. Should it be found upon examination to be as little chargeable with the graver fault of logical inaccuracy, it may perhaps, as far as it goes, bear a comparison with the former demonstrations.

That of **LEGENDRE** however certainly goes farther; as it proves the incommensurability, not only of the semi-circumference π , but also of its square π^2 . There is indeed every reason to conclude, short of absolute proof, that not only the square, but the cube, and all higher powers to infinity, are equally incommensurable. **Dr. BARROW**, in his *Mathematical Lectures*, has advanced some arguments in support of this opinion; but they have not the character of demonstration. The conjectures of **LEGENDRE** go even beyond this, implying the probability that the number π is not included among algebraical irrational quantities; in other words, that it cannot be the root of any algebraical equation having a finite number of terms with rational coefficients. But, he adds, a rigorous demonstration of this seems very difficult to find.

The short article on Infinite Series, which follows the demonstration in question, is designed to show the competency of ordinary algebra to effect, and with great ease, the summation of some extensive classes of infinite series which would at first sight seem necessarily to require the aid of definite integrals.

The algebraic principle employed in these summations

is one of extreme simplicity; being no other indeed than the principle of common subtraction. We have brought only two or three classes of series under its operation in the present short specimen; but it is easy to see that other classes might have been submitted to the same simple process of summation with equal success.

ANALYTICAL PROOF OF THE INCOMMENSURABILITY
OF THE CIRCUMFERENCE OF A CIRCLE.

(87.) THE theorem of DE MOIVRE, when expressed so as to comprehend fractional values of the exponent, may be written thus,

$$(\cos a + \sqrt{-1} \sin a)^A = (\cos Aa + \sqrt{-1} \sin Aa) 1^A.$$

But when integral values only of the exponent are to be admitted, then the formula will be complete without the factor 1^A . In the present investigation the theorem will be employed only in this restricted sense; and common multiplication alone is sufficient to establish its truth. It obviously furnishes the following four identities, namely,

$$\begin{aligned} (\cos a + \sqrt{-1} \sin a)^A &= \cos Aa + \sqrt{-1} \sin Aa \\ &= \cos aA + \sqrt{-1} \sin aA = (\cos a + \sqrt{-1} \sin a)^a, \end{aligned}$$

where A and a are both integers.

Now if any multiple of the circumference 2π , as

$k.2\pi$, could be a whole number, then putting $a = 2k\pi$, we should have, from the first and last of these identities, the equation

$$1^A = (\cos A + \sqrt{-1} \sin A)^{2k\pi},$$

or, substituting for $\cos A$, $\sin A$ their developments,

$$\cos A = 1 - \frac{A^2}{1.2} + \dots, \quad \sin A = A - \frac{A^3}{1.2.3} + \dots$$

the equation becomes

$$\begin{aligned} 1^A &= (1 + \sqrt{-1}.A - \frac{1}{2}A^2 + \dots)^{2k\pi} \\ &= \{1 + (\sqrt{-1} - \frac{1}{2}A + \dots)A\}^{2k\pi} \\ &= 1^{2k\pi} + 2k\pi.1^{2k\pi-1}(\sqrt{-1} - \frac{1}{2}A + \dots)A + \&c., \end{aligned}$$

that is, since $1^{2k\pi}$ is necessarily 1 by hypothesis, as also 1^A ,

$$1 = 1 + 2k\pi(\sqrt{-1} - \frac{1}{2}A + \dots)A + \frac{2k\pi(2k\pi-1)}{2}(\sqrt{-1} - \frac{1}{2}A + \dots)^2A^2 + \dots$$

consequently,

$$2k\pi\sqrt{-1}.A + 2k\pi(-\frac{1}{2} - \frac{2k\pi-1}{2})A^2 + \dots = 0;$$

or finally,

$$2k\pi\sqrt{-1}.A - 2(k\pi)^2A^2 + \dots = 0 \dots [1]$$

a series ascending according to the powers of A .

Now, by the hypothesis, the roots of this equation, that is to say, the values of A , being all integral, we must have

$$A = 0, = \pm 1, = \pm 2, \dots,$$

for the infinite number of the roots of the equation [1].

But these same values of A , to the exclusion of all others, are given by the equation

$$P \sin A\pi = 0,$$

or, developing the first member in powers of A , the same values are the roots of the equation.

$$P\pi A - \frac{P\pi^3}{1.2.3} A^3 + \frac{A\pi^5}{1.2.3.4.5} A^5 - \dots = 0,$$

or, which is the same thing, of the equation

$$P\pi A + 0A^2 - \frac{P\pi^3}{1.2.3} A^3 + 0A^4 + \frac{P\pi^5}{1.2.3.4.5} A^5 - \dots = 0 \dots [2]$$

Equations [1] and [2] must therefore be identical, each being produced from the same factors

$$\left. \begin{array}{l} A(A \pm 1)(A \pm 2)(A \pm 3) \dots \\ \text{or} \\ A(A^2 - 1)(A^2 - 2^2)(A^2 - 3^2) \dots \end{array} \right\} \dots [3]$$

Consequently,

$$2(k\pi)^2 = 0,$$

$$\therefore k\pi = 0;$$

so that the only admissible value of k is $k = 0$; therefore no multiple of the circumference can ever be a whole number. In fact, the product arising from [3] must evidently always have the coefficients of the even powers of A zero; so that we may infer, without reference to the development [2], that the coefficient of A^2 in [1] must be 0.

SUMMATION OF CERTAIN CLASSES OF INFINITE SERIES.

(88.) SINCE every fraction of the form

$$\frac{1}{n(n+p)(n+2p)\dots(n+mp)} \dots\dots [A]$$

is equal to $\frac{1}{mp}$ the difference between one of the form

$$\frac{1}{n(n+p)\dots[n+(m-1)p]} \dots\dots [B]$$

and another of the form

$$\frac{1}{(n+p)(n+2p)\dots(n+mp)} \dots\dots [C]$$

it follows that any series of fractions, having [A] for its general term, will be equal to $\frac{1}{mp}$ the difference between two series of equal extent, and of which the general term of the one is [B], and the general term of the other [C].

This simple and obvious principle is sufficient to guide us to the summation of a great variety of series; all those indeed of which [A] is the general type of the several terms, as we have already explained at length elsewhere.*

It is our intention in the present short paper to show that the principle may be extended to classes of series of much greater difficulty; those, namely, wherein the several denominators are formed not merely of the simple factors above, but of the squares of these, or of a combination of both kinds.

We have just observed, and the truth of the statement is immediately seen, that

$$A = \frac{1}{mp}(B - C),$$

consequently, squaring each side of this equality,

$$\begin{aligned} A^2 &= \frac{1}{m^2 p^2}(B - C)^2 \\ &= \frac{1}{m^2 p^2}(B^2 + C^2) - \frac{2}{m^2 p^2}BC. \end{aligned}$$

Hence any series of which A^2 is the general term will be equal to $\frac{1}{m^2 p^2}$ the sum of two series whose general terms are B^2 and C^2 respectively *minus* $\frac{2}{m^2 p^2}$ times a third series of which the general term is BC ; these series being all of equal extent, and commencing each at the same values of n and p .

Now it is obvious that the two series whose general terms are B^2 and C^2 , if extended to infinity, will together be equal to twice the series whose general term is B^2 *minus* the leading term in the same series; therefore, sup-

* Elements of Algebra, chap. viii.

posing $n = 1$ and calling the series whose terms are of the form B^2 , S' , and that whose terms are of the form A^2 , S the foregoing expression for A^2 will give for S the following value, namely,

$$S = \frac{1}{m^2 p^2} \left\{ 2S' - \frac{1}{1^2(1+p)^2(1+2p)^2 \dots [1+(m-1)p]^2} \right\} \dots [D]$$

minus $\frac{2}{m^2 p^2}$ the infinite series whose terms are of the form

$$\frac{1}{n(n+p)^2(n+2p)^2 \dots [n+(m-1)p]^2(n+mp)} \dots [D'];$$

But from the original relation which connects together A , B , and C , it is obvious that this fraction D' is equal to

$$\frac{1}{mp} \left\{ \frac{1}{n(n+p)^2(n+2p)^2 \dots [n+(m-1)p]^2} - \frac{1}{(n+p)^2(n+2p)^2 \dots [n+(m-1)p]^2(n+mp)} \right\}$$

so that the entire series whose terms are generally expressed by D' , will be equal to $\frac{1}{mp}$ the difference between two series whose corresponding terms are generally expressed by the two fractions within the braces.

It will be convenient however to regard the latter of these series, that is, the subtractive series, as originating at the term immediately preceding that at which it actually commences, and to correct the error after performing the subtraction, by adding to the remainder the leading term previously introduced. This leading term, n being equal to 1, is

$$\frac{1}{1^2(1+p)^2(1+2p)^2 \dots [1+(m-1)p]^2}:$$

Now the result of the subtraction spoken of is readily seen to be

$$-(m-1)pS',$$

and consequently, by introducing the above correction, the true difference between the two series will be expressed by

$$-\left\{(m-1)pS' - \frac{1}{1^2(1+p)^2(1+2p)^2 \dots [1+(m-1)p]^2}\right\}$$

It follows therefore that the value of the series S must be equal to the expression marked [D] together with $\frac{2}{m^2p^2}$ the expression just deduced; that is,

$$S = \frac{1}{m^2p^2} \left\{ 2S' - \frac{1}{1^2(1+p)^2(1+2p)^2 \dots [1+(m-1)p]^2} \right\} + \frac{2}{m^2p^2} \left\{ (m-1)pS' - \frac{1}{1^2(1+p)^2(1+2p)^2 \dots [1+(m-1)p]^2} \right\}$$

or, which is the same thing,

$$S = \frac{4m-2}{p^2} S' - \frac{3mp-2p+2}{m^2p^2} \times \left. \frac{1}{1^2(1+p)^2(1+2p)^2 \dots [1+(m-1)p]^2} \right\} \dots [a].$$

By this general formula any infinite series of the form

$$S = \frac{1}{1^2(1+p)^2(1+2p)^2 \dots} + \left. \frac{1}{(1+p)^2(1+2p)^2(1+3p)^2 \dots} + \&c. \right\} \dots [b]$$

may be readily summed, provided only that we previously know the summation of the series S' , whose terms differ from those of S by the absence of the final factor in the denominator of each.

Knowing therefore the value of the series

$$S' = \frac{1}{1^2} + \frac{1}{(1+p)^2} + \frac{1}{(1+2p)^2} + \&c. \dots [c],$$

we may, by successive applications of the formula $[a]$, proceed through all the intermediate summations up to the sum of $[b]$. And in those cases in which the value of $[c]$ is unknown, and its approximate summation required we may arrive at such approximation by first summing a few terms of the more rapid series $[b]$; and then by means of the formula $[a]$ gradually descending till we arrive at the proposed form $S' = [c]$.

If in the formula $[a]$ we assume $p = 1$, we shall have

$$S = \frac{1}{m^2} \left\{ \frac{2(2m-1)}{m} S' - \frac{3}{1^2 \cdot 2^2 \dots m^2} \right\} \dots [I].$$

And if $p = 2$

$$S = \frac{1}{2m^2} \left\{ (2m-1) S' - \frac{3m-1}{1^2 \cdot 3^2 \dots (2m-1)^2 \cdot 2} \right\} \dots [II].$$

To render these formulas available for the summation of infinite series of the forms

$$S = \frac{1}{1^2 \cdot 2^2 \cdot 3^2 \dots (1+m)^2} + \frac{1}{2^2 \cdot 3^2 \cdot 4^2 \dots (2+m)^2} + \dots$$

and

$$S = \frac{1}{1^2 \cdot 3^2 \cdot 5^2 \dots (1+2m)^2} + \frac{1}{3^2 \cdot 5^2 \cdot 7^2 \dots (3+2m)^2} + \dots ;$$

we must previously discover the sums of

$$S' = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

and

$$S' = \frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \dots$$

by some independent method. Analysts have been much occupied with the class of series to which these two forms belong. EULER especially has devoted considerable space and labour to their investigation; and it is to his researches principally that the enquiring student is recommended for ample information on this curious and interesting subject * None of the series belonging to the class referred to, where the denominators which succeed one another are powers of the same degree of an arithmetical series of numbers, are summable in finite numbers, although each sum may be expressed in a finite form; for it is a remarkable circumstance, that all these series are so connected with the circumference of a circle, that the numerical expression for this circumference, to radius unity, is invariably involved in the sum of the series. The sum of the first of the series S' is found to be $\frac{\pi^2}{6}$; that of the second $\frac{\pi^2}{8}$; the circumference to radius 1 being as usual represented by 2π .

These sums are very easily obtained as follows: Since the values of A which satisfy the condition

$$\sin A\pi = 0,$$

are all given by the series

$$A = 0, = \pm 1, = \pm 2, \&c.,$$

* *Analysin Infinitorum*, vol. i.

it follows that the roots of the equation,

$$\pi A - \frac{\pi^3}{1.2.3} A^3 + \frac{\pi^5}{1.2.3.4.5} A^5 - \&c. = 0,$$

whose first member is only the development of $\sin A\pi$, are all represented by the same numbers. Dividing this development by $A - 0$, or A , the depressed equation, containing the roots

$$\pm 1, \pm 2, \pm 3, \&c.,$$

is

$$\pi - \frac{\pi^3}{1.2.3} A^2 + \frac{\pi^5}{1.2.3.4.5} A^4 - \&c. = 0.$$

If we divide this by $A\pi^n$, the resulting equation

$$\left(\frac{1}{A^2}\right)^n - \frac{\pi^2}{1.2.3} \left(\frac{1}{A^2}\right)^{n-1} + \frac{\pi^4}{1.2.3.4.5} \left(\frac{1}{A^2}\right)^{n-2} - \&c. = 0,$$

will have its n roots the squares of the reciprocals of the roots of the preceding equation; that is, provided n be infinite. These roots are therefore

$$\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \&c.$$

And as the coefficient of the second term, taken with changed sign, is equal to the sum of the roots of every equation, it follows that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \&c. = \frac{\pi^2}{6}.$$

Again, the values of A in the expression

$$\cos A \frac{\pi}{2} = 0$$

are, exclusively,

$$A = \pm 1, \pm 3, \pm 5, \&c.$$

so that these numbers are the roots of the equation

$$1 - \frac{\frac{1}{2}\pi^2}{1 \cdot 2} A^2 + \frac{\frac{1}{6}\pi^4}{1 \cdot 2 \cdot 3 \cdot 4} A^4 - \&c. = 0;$$

consequently, dividing by A^{2n} , the roots of the equation

$$\left(\frac{1}{A^2}\right)^n - \frac{\frac{1}{2}\pi^2}{1 \cdot 2} \left(\frac{1}{A^2}\right)^{n-1} + \&c. = 0;$$

where n is infinite, are as follows, namely,

$$\frac{1}{1^2}, \frac{1}{3^2}, \frac{1}{5^2}, \&c.$$

therefore equating, as before, the sum of the roots with the coefficient of the second term, taken with the sign changed, we have

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \&c. = \frac{\pi^2}{8}.$$

and thus the fundamental series $[c]$, in the cases of $p = 1$ and $p = 2$, is known.

Referring now to the formula [I], we have, when

$$m = 1, S = \frac{\pi^2}{3} - 3 = \frac{1}{1^2 \cdot 2^2} + \frac{1}{2^2 \cdot 3^2} + \frac{1}{3^2 \cdot 4^2} + \&c.$$

$$m = 2, S = \frac{\pi^2}{4} - \frac{39}{16} = \frac{1}{1^2 \cdot 2^2 \cdot 3^2} + \frac{1}{2^2 \cdot 3^2 \cdot 4^2} + \frac{1}{3^2 \cdot 4^2 \cdot 5^2} + \&c.$$

$$m = 3, S = \frac{5\pi^2}{54} - \frac{197}{216} = \frac{1}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2} + \frac{1}{2^2 \cdot 3^2 \cdot 4^2 \cdot 5^2} + \&c.$$

and so on as far as we please.

In like manner from the formula [II], we have, when

$$m = 1, S = \frac{\pi^2}{16} - \frac{1}{2} = \frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \&c.$$

$$m = 2, S = \frac{3\pi^2}{256} - \frac{1}{9} = \frac{1}{1^2 \cdot 3^2 \cdot 5^2} + \frac{1}{3^2 \cdot 5^2 \cdot 7^2} + \frac{1}{5^2 \cdot 7^2 \cdot 9^2} + \&c.$$

$$m = 3, S = \frac{5\pi^2}{4608} - \frac{43}{4050} = \frac{1}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2} + \frac{1}{3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2} + \&c.$$

and so on to any extent.

(89.) By means of the same general relation

$$A = \frac{1}{mp} (B - C)$$

a variety of other series may be very expeditiously summed.

Let, for example, the form of A be

$$\frac{1}{n^2(n+p)(n+2p) \dots (n+mp)}$$

then, by the general relation just referred to, we may substitute for it the expression

$$\frac{1}{mp} \left\{ \frac{1}{n^k (n+p)(n+2p) \dots [n+(m-1)p]} - \frac{1}{n^{k-1} (n+p)(n+2p) \dots (n+mp)} \right\}$$

Now by the same relation we may, for the first term within the braces, substitute the following, namely,

$$\frac{1}{(m-1)p} \left\{ \frac{1}{n^k (n+p)(n+2p) \dots [n+(m-2)p]} - \frac{1}{n^{k-1} (n+p)(n+2p) \dots [n+(m-1)p]} \right\}$$

And for the first of these we may make a like substitution; so that, by proceeding in this manner, we shall at length arrive at

$$\frac{1}{2p} \left\{ \frac{1}{n^k (n+p)} - \frac{1}{n^{k-1} (n+p)(n+2p)} \right\}$$

and finally at

$$\frac{1}{p} \left\{ \frac{1}{n^k} - \frac{1}{n^{k-1} (n+p)} \right\}$$

whence a rule may be easily deduced for the summation of an infinite series of the proposed form, provided certain subordinate series can be summed.

As an illustration of this, suppose $n = 2$, and put for abridgment

$$\frac{1}{n^2} + \frac{1}{(n+p)^2} + \frac{1}{(n+2p)^2} + \&c. = s$$

$$\frac{1}{n(n+p)} + \frac{1}{(n+p)(n+2p)} + \frac{1}{(n+2p)(n+3p)} + \&c. = S_1$$

$$\frac{1}{n(n+p)(n+2p)} + \frac{1}{(n+p)(n+2p)(n+3p)} + \&c. = S_2$$

and generally,

$$\frac{1}{n(n+p) \dots (n+mp)} + \frac{1}{(n+p) \dots [n+(m+1)p]} + \&c. = S_m,$$

then we have the following rule for obtaining the sum of the infinite series

$$\frac{1}{n^2(n+p)(n+2p) \dots (n+mp)} +$$

$$\frac{1}{(n+p)^2(n+2p) \dots [n+(m+1)p]} + \&c.$$

From s take S_1 and divide the result by p ; from the quotient take S_2 and divide the result by $2p$; from this second quotient take S_3 and divide the result by $3p$, and so on till the divisor becomes mp . This divisor will furnish a quotient, which will be the sum of the proposed series.

As to the values of the subtractive quantities $S_1, S_2, S_3, \&c.$ they are at once obtained from the fundamental relation with which we set out. Thus

$$S_1 = \frac{1}{pn}$$

$$S_2 = \frac{1}{2pn(n+p)} = \frac{S_1}{2(n+p)}$$

$$S_3 = \frac{1}{3pn(n+p)(n+2p)} = \frac{2S_2}{3(n+2p)}$$

$$\vdots$$

$$S_m = \frac{(m-1)S_{m-1}}{m[n+(m-1)p]}.$$

As an example, let it be required to find the sum of the infinite series

$$\frac{1}{1^2 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{2^2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{3^2 \cdot 4 \cdot 5 \cdot 6} + \&c.$$

in which we have

$$n = 1, \quad p = 1, \quad m = 3.$$

The values of S_1, S_2, S_3 , are in this case

$$S_1 = 1, \quad S_2 = \frac{1}{4}, \quad S_3 = \frac{1}{18}.$$

Consequently, arranging these in a row, and prefixing the subtractive sign, the operation by the rule will be as follows, namely,

$$\begin{array}{rcccc}
 -1 & -\frac{1}{4} & -\frac{1}{18} & & \\
 \frac{\pi^2}{6} & \frac{\pi^2}{6} - 1 & \frac{\pi^2}{12} - \frac{5}{8} & \frac{\pi^2}{36} - \frac{49}{216} & = \text{Sum} \\
 \hline
 \frac{\pi^2}{6} - 1 & \frac{\pi^2}{6} - \frac{5}{4} & \frac{\pi^2}{12} - \frac{49}{72} & &
 \end{array}$$

If the series to be summed be

$$\frac{1}{1^2 \cdot 3 \cdot 5 \cdot 7} + \frac{1}{3^2 \cdot 5 \cdot 7 \cdot 9} + \&c.$$

in which $p = 2$, the process will be this, namely,

$$\begin{array}{r} -\frac{1}{2} \qquad -\frac{1}{12} \qquad -\frac{1}{90} \\ \frac{\pi^2}{8} \qquad \frac{\pi^2}{16} - \frac{1}{4} \qquad \frac{\pi^2}{64} - \frac{1}{12} \qquad \frac{\pi^2}{384} - \frac{17}{1080} = \text{Sum} \\ \hline \frac{\pi^2}{8} - \frac{1}{2} \qquad \frac{\pi^2}{16} - \frac{1}{3} \qquad \frac{\pi^2}{64} - \frac{17}{180} \end{array}$$

By imitating the steps of the preceding investigation, it would be easy to deduce a rule for the summation of the infinite series

$$\frac{1}{n(n+p) \dots (n+mp)^2} + \frac{1}{(n+p) \dots [n+(m+1)p]^2} + \&c.$$

This rule would differ from the preceding one in the following particulars, namely, instead of S_1 , S_2 , S_3 , &c. we should have to employ

$$\begin{array}{l} S_1 + \frac{1}{n^2} \\ S_2 + \frac{1}{n(n+p)^2} \\ S_3 + \frac{1}{n(n+p)(n+2p)^2} \\ \vdots \\ S_m + \frac{1}{n(n+p) \dots [n+(m-1)p]^2} \end{array}$$

and these, instead of being subtracted, as before, from the

several quantities placed under them, must themselves be diminished by those quantities.

It is plain that when n and p are each unity, as in the first of the foregoing examples, the values which here supply the place of S_1, S_2, S_3 , &c., will be the doubles of these latter quantities.

As an example let the series

$$\frac{1}{1 \cdot 2 \cdot 3 \cdot 4^2} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5^2} + \&c.$$

be proposed.

In this case

$$n = 1, \quad p = 1, \quad m = 3,$$

$$\text{and } 2 S_1 = 2, \quad 2 S_2 = \frac{1}{2}, \quad 2 S_3 = \frac{1}{9};$$

consequently, the operation is as follows:

2	$\frac{1}{2}$	$\frac{1}{9}$	
$\frac{\pi^2}{6}$	$2 - \frac{\pi^2}{6}$	$\frac{\pi^2}{12} - \frac{3}{4}$	$\frac{31}{108} - \frac{\pi^2}{36} = \text{Sum}$
$2 - \frac{\pi^2}{6}$	$\frac{\pi^2}{6} - \frac{3}{2}$	$\frac{31}{36} - \frac{\pi^2}{12}$	

By comparing the several steps of this process with those in the work of the first example, we are led to conclude that the aggregate of two series like this last and the one referred to will always be accurately expressed in a finite fraction, provided the factors in the denominators be even in number, and that the difference of the two series will also be a finite fraction if the number of these factors be odd. In all other cases the sum and difference will involve π^2 . Thus the difference between the two series

$$\frac{1}{1^2 \cdot 2 \cdot 3} + \frac{1}{2^2 \cdot 3 \cdot 4} + \frac{1}{3^2 \cdot 4 \cdot 5} + \&c. = \frac{\pi^2}{12} - \frac{5}{8},$$

and

$$\frac{1}{1 \cdot 2 \cdot 3^2} + \frac{1}{2 \cdot 3 \cdot 4^2} + \frac{1}{3 \cdot 4 \cdot 5^2} + \&c. = \frac{\pi^2}{12} - \frac{3}{4}$$

is $\frac{1}{8}$; so that, actually subtracting and dividing by 2, we have

$$\frac{1}{1^2 \cdot 2 \cdot 3^2} + \frac{1}{2^2 \cdot 3 \cdot 4^2} + \frac{1}{3^2 \cdot 4 \cdot 5^2} + \&c. = \frac{1}{16}.$$

Therefore this last series is the square of the series

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} = \frac{1}{4}.$$

It thus appears that every infinite series of the form

$$\frac{1}{1^2 \cdot 2 \cdot 3 \dots (m-1) m^2} + \frac{1}{2^2 \cdot 3 \cdot 4 \dots m(m+1)^2} + \&c.$$

is accurately summable whenever the number of factors in each denominator is odd. And it is easy to see that the same is true of every infinite series of the form

$$\frac{1}{1^2 \cdot 3 \cdot 5 \dots (m-2) m^2} + \frac{1}{3^2 \cdot 5 \cdot 7 \dots m(m+2)^2} + \&c.$$

But when in either case the number of factors is even, then the sum will always involve π^2 .*

* For other methods of summing infinite series, whose forms are the same as those considered in the former part of this article, the student is referred to two very ingenious papers in the Appendix to the *Ladies' Diary* for 1836 and 1837: the first by Mr. Woolhouse, and the second by Mr. RUTHERFORD of the Royal Military Academy.

DISSERTATION IV.

ON SOME PRACTICAL IMPROVEMENTS IN THE THEOREM
OF STURM, WITH NUMERICAL ILLUSTRATIONS.

DISSERTATION IV.

ON THE THEOREM OF STURM.*

(90.) THE THEOREM OF STURM, a detailed account of which we have elsewhere given, is remarkable in two respects. First, because of the unerring certainty with which it accomplishes an object of paramount importance in algebraical calculation; and which, by the former tentative methods, was often of very doubtful and difficult attainment; and secondly, because of the simple and elementary character of the operations involved in its practical application, these operations being none other than those employed in the ordinary method of finding the greatest common measure of two algebraical expressions.

The two expressions which, in the theorem of STURM, are thus operated upon, are the original polynomial and its first derived function or differential coefficient. Of these a common measure actually exists only when the proposed equation has equal roots; so that when, as

* It will be presumed, that the reader of this dissertation will already have made himself acquainted with the nature and objects of Sturm's theorem. This theorem was first made known to the British student in the *Theory and Solution of Equations*, by the author of the present work.

usually happens, such is not the case, the operation will not terminate till we arrive at a constant remainder; that is, at a remainder purely numerical.

Whenever, therefore, the proposed equation is of an elevated order, the process will not only be long, but in consequence of the magnitude of the numerical coefficients, it may also become arithmetically laborious. But whatever method be proposed for effecting the objects which STURM's theorem so successfully attains, we may always expect the labour to increase with these same circumstances, namely, the degree of the equation and the magnitude of the coefficients. And indeed there is not an operation in common arithmetic, however simple in itself, that might not become painfully laborious from the magnitude of the numbers submitted to it.

(91.) In estimating the value of STURM's theorem, in comparison with other methods, we ought not however to confine our attention merely to the amount of numerical labour involved in each: it must be remembered, as peculiarly characteristic of STURM's process, that every step of this labour is effective, and does really advance us, and by a direct path, towards the end in view. In other methods, which, except the intolerably laborious one of LAGRANGE, are all more or less tentative and conjectural, we must work in doubt and uncertainty till some decided indication presents itself, either that we are on the right track, or that our work has been thus far fruitless, and must be renewed in another direction; and a very discouraging circumstance is, that we can seldom foresee at the outset to how remote a point in our progress this indication may be deferred.

The practical transformations involved in the methods of BUDAN and FOURIER, are, when viewed individually, comparatively easy, much easier in general than the steps

of STURM's theorem. But the great drawback to the usefulness of these methods is, that the information sought from them, except, indeed, in particular cases, is not of that precise and determinate character which is requisite to enable us to pronounce with confidence upon the constitution of the equation, which it is their object to analyze.

By successive applications of the various criteria proposed by these authors, we can no doubt reach, in most cases, a degree of probability as to the nature of the roots of an equation, amounting almost to absolute certainty; yet, after exhausting all the expedients which they recommend, there *may* still remain unsatisfied scruples in the mind of the computer, of sufficient influence to deter him from drawing a positive conclusion without either trying some independent means of verification, or else entering upon a supplementary calculation, involving almost the very same amount of numerical labour as that which, by the method of STURM, suffices for the complete solution of the problem.

The only labour worth mentioning which STURM's theorem requires is that implied in finding the greatest common measure of two algebraical expressions, the one, as already stated, being the polynomial which forms the significant side of the proposed equation, and the other being its first derived function; and this is the very process which, after having executed the operations peculiar to FOURIER's method, it is sometimes necessary to go through before we can safely interpret the results to which those operations may conduct us. FOURIER, fully aware of the imperfection or inadequacy of his method, does himself recommend this supplementary process to be applied in all those doubtful cases where, within a given interval, the roots may be either very nearly equal or else imaginary; and he uniformly regards this determination of

the common measure as essential to the completion of his own method, and accordingly incorporates it in his rule for separating the roots of an equation. (See *Analyse des Equations*, p. 124.)

Innumerable equations may no doubt be contrived, so that the rules of BUDAN and FOURIER will be found amply sufficient to make known their constitution without resorting to the process for the common measure. But no one can affirm, *à priori*, that, propose whatever equation you may, the directions given by these authors are sufficient to lead us unfailingly to the knowledge of the number and nature of its roots. This arises from the circumstance, that the marks or indications, which are shown always to accompany certain relations among the roots, *may* present themselves without being thus accompanied: if the relations really exist, there, unquestionably, will be the marks; but these latter may occur in the absence of such relations: the theorems offered for our guidance are not convertible; and they thus have the same incompleteness, and leave on the mind the same indecision, that is known to characterize all the older rules for discovering the composition of an equation.

The method proposed by STURM is free from all these perplexities. It completely and satisfactorily solves the great algebraical problem of determining the number and nature of the real roots of a numerical equation, and all that is necessary to its perfection is, that some means be contrived for conducting the numerical application of it in such a manner as to render the arithmetical operations the fewest possible.

The complete attainment of this object we cannot but regard as a matter of the first importance in the present state of algebra. It appears to us that every future improvement in the solution of numerical equations must originate in some practical reduction of the arithmetical

work implied in STURM's process; and that no simpler principle, of equal generality, for the analysis of a numerical equation, is ever likely to be discovered.

We ground this prediction upon the indisputable fact, that the principle of STURM is virtually that which must always either precede as preparatory, or follow as supplementary, every process for analyzing an equation having roots either accurately equal, or nearly equal to one another.

In cases such as these—and, it must be remembered, they are the cases comprehending all the difficulties of the subject—the operation for the common measure is known to be indispensable; and this, as already remarked, is the very operation which constitutes the whole of STURM's method.

It will be our main purpose in the present dissertation so to conduct the steps of this operation as to reduce the work of their successive derivation to the smallest amount, and then to contrive some safe means of restricting the numerical remainders thus derived within manageable bounds.

In the early stages of the process we shall never have occasion for any such abbreviating contrivances; but as the operation draws to a close, the numerical results of the final steps are generally found to increase, in the higher equations, at a very rapid rate; so that without the aid of some controlling principle, the practicability of the method would soon cease, merely on account of the magnitude of the numbers submitted to it.

Nevertheless, by a judicious arrangement of the elements of the calculation, much saving both of space and labour may be effected, without the aid of any abbreviating expedient whatever; and thus the method may be shown to be readily applicable to equations that have been affirmed to lie barely within its reach; we mean equations of the fourth and fifth degrees.

To devise such an arrangement of the calculation will be our first and more immediate object; we shall afterwards show how to check the advance of the numerical coefficients in the higher equations, and shall exhibit the efficacy of the method in the analysis of complete equations of the sixth and seventh degrees.

(92.) The problem that is now to occupy us therefore is this, viz. to effect, with the least possible expenditure of numerical labour, the division of a rational polynomial in x , of the degree n , by another of the degree $n - 1$, with the view of discovering the polynomial of the degree $n - 2$, which forms the *remainder*, it being understood that we are at liberty to multiply the dividend by any positive numerical factor that will preclude the entrance of fractions into this remainder.*

Let the dividend be

$$ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + \dots [A],$$

and the divisor

$$a'x^{n-1} + b'x^{n-2} + c'x^{n-3} + d'x^{n-4} + \dots [B].$$

The remainder, of degree $n - 2$, which it is our object to find, will obviously present itself after two terms of the quotient have been obtained. The first of these two terms, when incorporated with the divisor, destroys the first term of the dividend, and furnishes a remainder of the degree $n - 1$, the leading term in which is in like manner destroyed by means of the second term of the quotient: the second remainder thence resulting is that sought. But, as

* The method of conducting the operation when the two proposed polynomials differ in degree by two or more units, will be considered afterwards.

this remainder is to be without fractions, it is requisite that *both* terms of the quotient be without fractions. That the first of these terms may have no fraction, it is obviously merely necessary to multiply the dividend by a' ; but then the first remainder, adverted to above, having for its leading term $(a'b - ab')x^{n-1}$, would necessitate the entrance of a fraction into the second term of the quotient, namely, the fraction $\frac{a'b - ab'}{a'}$, with which fraction therefore the next remainder would be affected in all its terms.

It is generally necessary therefore to multiply the dividend, not merely by a' , but by a'^2 , after which it is easy to see that the quotient will be $a'ax + (a'b - ab')$. Multiplying then each of these terms by the divisor, and writing down the several partial products with changed signs, instead of actually subtracting them from the dividend, the work under this dividend will arrange itself as follows, the several remainders being really obtained by algebraical addition :

$$\begin{array}{r}
 a'^2 \cdot ax^n + a'^2 \cdot bx^{n-1} + a'^2 \cdot cx^{n-2} + a'^2 \cdot dx^{n-3} + \dots \\
 -a'a \cdot a'x^n \quad -a'a \cdot b'x^{n-1} - a'a \cdot c'x^{n-2} - a'a \cdot d'x^{n-3} - \dots \\
 \hline
 0 \quad -(a'b - ab')a'x^{n-1} - (a'b - ab')b'x^{n-2} - (a'b - ab')c'x^{n-3} - \dots \\
 \hline
 \qquad \qquad \qquad 0 \qquad \qquad + \qquad \qquad a''x^{n-2} \qquad + \qquad \qquad b''x^{n-3} + \dots
 \end{array}$$

..[C].

From this mode of arranging the work it is easy to discover what functions the coefficients a'' , b'' , &c., in the sought remainder, are of the coefficients in the original expressions. But, in order that a'' , b'' , &c., may be derived from the proposed coefficients in the most expeditious manner, the work should be conducted conformably to the following type of the operations :

<i>Multipliers.</i>	$\begin{array}{r} a'-b' - c' - d' - \dots \\ \times \\ a+b + c + d + \dots \\ \hline -Ab' - Ac' - Ad' - \dots \\ -Bc' - Bd' - Be' - \dots \\ Cc + Cd + Ce + \dots \\ \hline a'' + b'' + c'' + \dots \end{array}$
$A = (a'b - ab')$	$-Ab' - Ac' - Ad' - \dots$
$B = aa'$	$-Bc' - Bd' - Be' - \dots$
$C = a^2$	$Cc + Cd + Ce + \dots$

The complete remainder, to which these coefficients belong, is

$$a''x^{n-2} + b''x^{n-3} + c''x^{n-4} + \dots$$

In STURM's theorem, the signs of the terms of each remainder are to be changed before that remainder is employed in deducing, as above, the next subsequent one; so that the two polynomials concerned in the immediately following step of the process are as follow, namely, for the dividend,

$$a'x^{n-1} + b'x^{n-2} + c'x^{n-3} + d'x^{n-4} + \dots;$$

and for the divisor,

$$-a''x^{n-2} - b''x^{n-3} - c''x^{n-4} - d''x^{n-5} - \dots$$

Agreeably to the foregoing type of the work involved in each step, this last polynomial is to have all its signs, after the first, changed; the coefficients only are to be written down, those of the preceding polynomial, with their proper signs, are to be placed under them, and then the operations in the model are to be performed. Uniting, therefore, the consecutive steps, the following will be a general type of the entire calculation :

$$\begin{array}{rcl}
 & a' - b' - c' - d' - \dots & \\
 & \times & \\
 & a + b + c + d + \dots & \\
 \hline
 A = a'b - ab' & \left[\begin{array}{l} -Ab' - Ac' - Ad' - \dots \\ -Bc' - Bd' - Be' - \dots \\ Cc + Cd + Ce + \dots \end{array} \right. & \\
 B = a\alpha' & & \\
 C = a'^2 & & \\
 \hline
 & -a'' - b'' - c'' - \dots = 1st\ rem. with signs changed. & \\
 & \times & \\
 & -a'' + b'' + c'' + \dots & \\
 & a' + b' + c' + \dots & \\
 \hline
 A' = -a''b' + a'b'' & \left[\begin{array}{l} A'b'' + A'c'' + \dots \\ B'c'' + B'd'' + \dots \\ C'c' + C'd' + \dots \end{array} \right. & \\
 B' = -a'a'' & & \\
 C' = a'^2 & & \\
 \hline
 & -a''' - b''' - \dots = 2d\ rem. with signs changed. & \\
 & \times & \\
 & -a''' + b''' + \dots & \\
 & -a'' - b'' - \dots & \\
 \hline
 A'' = a'''b'' - a''b''' & \left[\begin{array}{l} A''b''' + \dots \\ B''c''' + \dots \\ -C''c'' - \dots \end{array} \right. & \\
 B'' = a''a''' & & \\
 C'' = a'''^2 & & \\
 \hline
 & -a'''' - \dots = 3d\ rem. with signs changed. & \\
 & \times & \\
 & & \\
 & \&c. \ \&c. &
 \end{array}$$

And, consequently, STURM's series of functions are,

$$X = ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + ex^{n-4} + \dots$$

$$X_1 = a'x^{n-1} + b'x^{n-2} + c'x^{n-3} + d'x^{n-4} + \dots$$

$$X_2 = -a''x^{n-2} - b''x^{n-3} - c''x^{n-4} - \dots$$

$$X_3 = -a'''x^{n-3} - b'''x^{n-4} - \dots$$

$$X_4 = -a''''x^{n-4} - \dots$$

&c. &c.

It is scarcely necessary to state that the cross, interposed between the first two terms of the dividend and divisor at the head of each step of the work as exhibited above, is inserted merely for the purpose of pointing out the mode of forming the first of the three multipliers used in that step. These three multipliers are arranged in the first column of the work, and the completion of this column is to be the first object in each step. The formation of the second and third multipliers is so simple and obvious, that, with respect to them, any such contrivance as that adopted in the case of the first multiplier, to assist the memory, would be superfluous.

(93.) There are two precepts concerning these multipliers, and the results furnished by them, which, if attended to by the computer, will often save him some portion of the numerical labour: there will indeed occur, occasionally, other opportunities for abridging the work; these we shall point out in connexion with the arithmetical process itself, whenever they present themselves in our practical examples. The two precepts, however, to which we now refer, indicating the more important of the means by which a reduction of labour may be effected, deserve to be distinctly stated and remembered,—they are as follow :

1. The multipliers will sometimes have a common measure, discoverable at sight: when this happens the common factor should be suppressed, and the multipliers, thus reduced, should be employed instead.

It is easy to see that, when either the first term of the dividend, or the second term of the divisor in any step, has a factor in common with the leading term of the divisor, such reduction may *always* be effected.

When the second coefficient of the divisor is zero, then it is obvious that the leading coefficient of the divisor will be a factor common to the three multipliers: it is therefore

to be expunged from them, or to be suppressed rather in writing these multipliers down.

These simplifications in the side-column of multipliers being always effected when practicable, the several remainders will thus become reduced in numerical value; and will therefore be more easily manageable in the subsequent steps of the process.

But they will not always be reduced, by these means, to their lowest possible amount; since the several coefficients, belonging to any remainder, may, even after the reductions here recommended, still involve a common factor; so that with a view to the utmost degree of simplicity, it becomes further necessary to examine the final result of each step; that is,

2. At the close of each step examine whether or not a common factor exist throughout the terms of the result; if such a factor be found to enter expunge it; and then employ the more simple form in the next step.

If the numerical operations be carried on in imitation of the preceding model, and the simplifications recommended in these precepts be always introduced, when obviously practicable, the labour attending the application of STURM's theorem will be reduced to nearly the least amount possible.

(94.) Still, however, the utmost degree of simplicity will not even yet have been reached; for the first step in the process may be completed without the aid of the model, and with superior facility; so that to conduct the entire operation with the least expenditure of time and labour, it will be necessary to defer the use of the guiding model till after the first step of the work has been completed by the shorter method here adverted to. But we think it better to postpone this final improvement, as also certain other abbreviating contrivances, for consideration hereafter; and to work for the present entirely by the

model, in order that the student may become previously familiar with the general law by which each step is uniformly derived from that which precedes it. Let us now proceed to examples.

1. Required the number and situations of the real roots of the equation

$$x^4 - 3x^3 - 2x^2 + x - 3 = 0.$$

In this example the first dividend is the polynomial

$$x^4 - 3x^3 - 2x^2 + x - 3,$$

and the corresponding divisor is the first derived function of it, namely,

$$4x^3 - 9x^2 - 4x + 1.$$

Hence the work, conformably to the general type, is as follows:

$$\begin{array}{r}
 4+ \quad 9+4-1 \\
 \times \\
 1- \quad 3-2+1-3 \\
 \hline
 -3 \quad -27-12+ \quad 3 \\
 4 \quad \quad 16- \quad 4 \\
 16 \quad -32+16-48 \\
 \hline
 43+0 \quad +45 \\
 \hline
 43-0 \quad -45 \\
 \times \\
 4-9 \quad - \quad 4+1 \\
 \hline
 -9 \quad \quad 0+405 \\
 4 \quad -180 \\
 43 \quad -172 \quad +43 \\
 \hline
 352-448 \\
 \text{or, } \div 4 \times 8 \\
 11-14.
 \end{array}$$

One more step would bring us to the final remainder; but, as the *sign* of this final remainder is all that we require to know respecting it, the closing step of the operation may always be dispensed with, and the required sign ascertained more easily in another manner; as we shall presently show.

(95.) The value of the two precepts previously laid down is clearly seen in the second of the above steps. By the first precept the numbers forming the column of multipliers are considerably reduced by the suppression of the common factor 43; and by the second precept the result of that step is still further reduced by expunging the common factor 32.

Omitting the constant remainder, or the final function, STURM's functions are therefore as follow :

$$X = x^4 - 3x^3 - 2x^2 + x - 3$$

$$X_1 = 4x^3 - 9x^2 - 4x + 1$$

$$X_2 = 43x^2 + 45$$

$$X_3 = 11x - 14.$$

Now, although we have deduced the function X_3 , for the purpose of illustrating both the general method of derivation, and the precepts for simplification, yet there is no occasion for this function in determining the nature of the roots of the proposed equation. The first step really completes the operation; for it is seen in a moment, that the roots of the equation $43x^2 + 45 = 0$, furnished by this step, are imaginary; and STURM proves that as soon as we arrive at a function of this kind the subsequent steps

of the work are superfluous. (See *Theory of Equations*, p. 146, also Note ii. at the end of this Dissertation.)

Hence, employing only the functions

$$X \quad X_1 \quad X_2$$

we have, for $x = -\infty$, $+$ $-$ $+$

..... $x = +\infty$, $+$ $+$ $+$

therefore, as two variations only are lost between $-\infty$ and $+\infty$, we infer that there are but two real roots.

To ascertain the situations of these, substitute 0, 1, 2, 3, &c., for x ; and then 0, -1, -2, -3, &c., for it is plain from the sign of the last term in x that one of these real roots will be positive and the other negative. We need not make these substitutions in all three of the functions, but merely in the first X ; since a change of sign in this is all that can mark the passage of a single root. The positive root is thus readily found to lie between 3 and 4; and the negative root between -1 and -2.

In the present example the functions X_1 and X_2 happen to be unnecessary; the final one X_3 , or at least the sign of it, which is all we can ever require to know about the final function, might have been easily ascertained from considering that the remainder, arising from the division of a polynomial by an expression of the form $x \pm a$, is obtained by the mere substitution of $\mp a$ for x in that polynomial, (*Theory of Equations*, Art. 10;) so that in the example before us the remainder is found by putting $\frac{14}{11}$ for x in $43x^2 + 45$; and we see at a glance that this remainder must be *plus*; hence, $X_3 = -$. And in this manner should the sign of the final remainder always be determined.

This method of obtaining the sign of the final remainder is also suggested from the known property of STURM'S functions, that when any one vanishes, for a particular

value of x , the immediately preceding and following functions take opposite signs for that same value of x . The function immediately preceding the last vanishes for $x = \mp a$; hence the sign of the *last* function must be opposite to that which the *last but two* takes for $x = \mp a$.

2. Required the number and situations of the real roots of the equation

$$x^4 - 2x^3 - 7x^2 + 10x + 10 = 0.$$

Here $X_1 = 2x^3 - 3x^2 - 7x + 5$.

$$\begin{array}{r}
 2+3 \quad +7 \quad -5 \\
 \times \\
 1-2 \quad -7+10+10 \\
 \hline
 -1 \left| \begin{array}{r} -3 \quad -7 \quad +5 \\ 2 \quad 14-10 \\ 4 \quad -28+40+40 \end{array} \right. \\
 \hline
 17-23-45 \\
 \hline
 17+23+45 \\
 \times \\
 2-3-7+5 \\
 \hline
 -5 \left| \begin{array}{r} -115 \quad -225 \\ 34 \quad 1530 \\ 289 \quad -2023+1445 \end{array} \right. \\
 \hline
 608-1220 \\
 \text{or, } \div 4 \\
 152 \quad -305.
 \end{array}$$

Consequently

$$X = x^4 - 2x^3 - 7x^2 + 10x + 10$$

$$X_1 = 2x^3 - 3x^2 - 7x + 5$$

$$X_2 = 17x^2 - 23x - 45$$

$$X_3 = 152x - 305$$

$$\therefore X_4 = +$$

As the leading terms of these functions are all *plus*, the roots of the equation are all real; their situations have already been determined in the *Theory of Equations*, p. 150.

3. Required the number and situations of the real roots of the equation

$$x^4 + x^3 - 15x^2 - 19x - 3 = 0.$$

The leading functions are

$$x^4 + x^3 - 15x^2 - 19x - 3,$$

and

$$4x^3 + 3x^2 - 30x - 19,$$

and the work is therefore as follows:

$$\begin{array}{r}
 \begin{array}{r}
 4-3 \quad +30+19 \\
 \times \\
 1+1 \quad -15-19-3 \\
 \hline
 1 \quad -3 \quad +30+19 \\
 4 \quad 120 \quad +76 \\
 16 \quad -240-304-48 \\
 \hline
 \quad 123+198+29 \\
 \hline
 \quad 123-198-29 \\
 \times \\
 \quad 4+3 \quad -30-19 \\
 \hline
 -141 \quad 27918 \quad +4089 \\
 164 \quad -4756 \\
 41^2 \times 3 \quad -151290 \quad -95817 \\
 \hline
 \quad 128128 \quad +91728
 \end{array}
 \end{array}$$

Therefore the functions are

$$X = x^4 + x^3 - 15x^2 - 19x - 3$$

$$X_1 = 4x^3 + 3x^2 - 30x - 19$$

$$X_2 = 123x^2 + 198x + 29$$

$$X_3 = 128128x + 91728$$

$$\therefore X_4 = +$$

As the leading signs of these functions are all plus, we infer that the roots are all real; moreover, as for $x = 0$ there will be one variation, shown by the signs of the final terms, there is only one positive root. This is found from the single function X , without the aid of the others, to lie between 3 and 4. To find the situations of the negative roots, it will be convenient to convert them into positive; that is, to change the signs of the alternate terms in these functions, writing them thus:

$$X = x^4 - x^3 - 15x^2 + 19x - 3$$

$$X_1 = 4x^3 - 3x^2 - 30x + 19$$

$$X_2 = 123x^2 - 198x + 29$$

$$X_3 = 128128x - 91728$$

$$X_4 = +$$

$$X \quad X_1 \quad X_2 \quad X_3 \quad X_4$$

for $x = 0 \quad \dots - \quad + \quad + \quad - \quad +$ *three variations*

$$= 1 \quad \dots + \quad - \quad - \quad + \quad +$$
 two

$$= 2 \quad \dots - \quad - \quad + \quad + \quad +$$
 one variation

$$= 3 \quad \dots - \quad + \quad + \quad + \quad +$$
 one

$$= 4 \quad \dots + \quad + \quad + \quad + \quad +$$
 no variation.

Hence the negative roots of the proposed equation are situated between 0 and -1 ; -1 and -2 ; and -3 and -4 .

4. Required the number and situations of the real roots of the equation

$$x^5 + 5x^3 - 10x^2 + 5x - 1 = 0.$$

The function X_5 , when disencumbered of the common factor 5, is in this case

$$x^4 + 0x^3 + 3x^2 - 4x + 1.$$

$$\begin{array}{r} 1+0-3+4-1 \\ \times \\ 1+0+5-10+5-1 \end{array}$$

$$\begin{array}{r} 0 \\ 1 \\ 1 \end{array} \left[\begin{array}{r} -3+4-1 \\ 5-10+5-1 \end{array} \right]$$

$$\begin{array}{r} -2+6-4+1 \end{array}$$

$$\begin{array}{r} -2-6+4-1 \end{array}$$

$$\begin{array}{r} \times \\ 1+0+3-4+1 \end{array}$$

$$\begin{array}{r} -3 \\ -1 \\ 2 \end{array} \left[\begin{array}{r} 18-12+3 \\ -4+1 \\ 6-8+2 \end{array} \right]$$

$$\begin{array}{r} -20+19-5 \end{array}$$

$$\begin{array}{r} -20-19+5 \end{array}$$

$$\begin{array}{r} \times \\ -2+6-4+1 \end{array}$$

$$\begin{array}{r} -41 \\ 20 \\ 200 \end{array} \left[\begin{array}{r} 779-205 \\ 100 \\ -800+200 \end{array} \right]$$

$$\begin{array}{r} -79+5 \end{array}$$

Hence the functions are

$$X = x^5 + 5x^3 - 10x^2 + 5x - 1$$

$$X_1 = x^4 + 3x^2 - 4x + 1$$

$$X_2 = -2x^3 + 6x^2 - 4x + 1$$

$$X_3 = -20x^2 + 19x - 5$$

$$X_4 = -79x + 5$$

$$\therefore X_5 = +$$

(96.) In this example the last two functions have been deduced merely for the sake of further illustrating the mode of derivation. They are not necessary to the solution of the problem; for since $4(5 \times 20) > 19^2$ the roots of $X_3 = 0$ are imaginary.

Employing therefore only the functions

$$X \quad X_1 \quad X_2 \quad X_3$$

we have, for $x = -\infty$, $- \quad + \quad + \quad -$ *two variations*

..... $x = +\infty$, $+ \quad + \quad - \quad -$ *one.*

Hence there is but one real root; and, since the last term of X is negative, this root must be positive. A superior limit to this positive root will be found by dividing the greatest negative coefficient, 10, by the greatest positive one, 5, which precedes it, and then adding 1 to the quotient. (*Theory of Equations*, Proposition IV. Chap. III.) This superior limit will therefore be 3; so that we need try only the numbers from 0 to 3, in the single function X . We thus find that $x = 1$ is the real root.

5. Required the number and situations of the real roots of the equation

$$x^5 + 3x^4 + 2x^3 - 3x^2 - 2x - 2 = 0.$$

The function X_1 is

$$\begin{array}{r}
 5x^4 + 12x^3 + 6x^2 - 6x - 2 \\
 \begin{array}{r}
 5-12 \quad -6+6+2 \\
 \times \\
 1+3 \quad +2 \quad -3 \quad -2-2
 \end{array} \\
 \hline
 3 \quad -36-18+18+6 \\
 5 \quad -30+30+10 \\
 25 \quad 50-75-50-50 \\
 \hline
 16+63+22+44 \\
 \hline
 16-63-22-44 \\
 \times \\
 5+12 \quad +6 \quad -6-2 \\
 -123 \quad 7749+2706+5412 \\
 80 \quad -1760-3520 \\
 256 \quad 1536-1536 \quad -512 \\
 \hline
 -7525+2350-4900 \\
 \text{or, } \div 5^3 \\
 -301+94-196.
 \end{array}$$

The operation terminates here, as in the preceding example, because $4(301 \times 196) > 94^2$. The functions, as far as necessary, are therefore

$$X = x^5 + 3x^4 + 2x^3 - 3x^2 - 2x - 2$$

$$X_1 = 5x^4 + 12x^3 + 6x^2 - 6x - 2$$

$$X_2 = 16x^3 + 63x^2 + 22x + 44$$

$$X_3 = -301x^2 + 94x - 196$$

These give, for $x = -\infty$, $- + -$ two variations

$x = +\infty$, $+ + +$ one variation.

Hence there is but one real root, which must be positive, because the last term of X is negative. Moreover, a superior limit to this root, determined as in last example, is 2; so that no number beyond 2 need be substituted in X . Putting $x = 0$ in X , the result is *minus*; putting $x = 1$, the result is still *minus*; we infer therefore that the root is between 1 and 2.

(97.) If we had had occasion to extend our computation up to X_4 in this example, the preparation for the additional step would have been

$$\begin{array}{r}
 -301-94+196 \\
 \quad \times \\
 \quad 16+63+22+44 \\
 \hline
 -20467 \\
 -4816 \\
 301^2
 \end{array}$$

But as 94, 196, 22, and 44 have the common factor 2, we should have suppressed that factor; and, preserving the side column of multipliers as it is, should have employed the two horizontal rows, as if they had been written thus,

$$\begin{array}{r}
 -301-47+98 \\
 16+63+11+22
 \end{array}$$

and then have proceeded as usual.

Whenever at the commencement of any step the numbers in the first horizontal row, beginning with the *second* of those numbers, each involve a factor common also to the numbers in the second row, beginning with the *third*, as in the instance before us, the common factor may always be neglected, and thus the subsequent work shortened.

We shall give an example in which this simplification may be introduced.

6. Required the number and situations of the real roots of the equation

$$x^4 - 14x^3 + 66x^2 - 123x + 90 = 0.$$

In this example the reduction referred to above may be applied to the first step of the work, the numbers being divisible by 3. We shall therefore commence the step as usual; and place the results of this division over the original numbers as follows:

$$\begin{array}{r}
 \begin{array}{ccccccc}
 & 14 & & 44 & & 41 & \\
 4 & +42 & -132 & +123 & & & \\
 \times & & 22 & & 41 & & 30 \\
 1 & -14 & +66 & -123 & +90 & &
 \end{array} \\
 -7 \left[\begin{array}{l} -98 + 308 - 287 \\ -88 + 82 \\ 176 - 328 + 240 \end{array} \right. \\
 \hline
 \begin{array}{l} 10 - 62 + 47 \\ 10 + 62 - 47 \\ \times \\ 4 - 42 + 132 - 123 \end{array} \\
 -43 \left[\begin{array}{l} -2666 + 2021 \\ -470 \\ 3300 - 3075 \end{array} \right. \\
 10 \left[\begin{array}{l} -164 + 1054 \\ -82 + 527 \end{array} \right. \\
 5^2 \left[\begin{array}{l} -82 + 527 \end{array} \right. \\
 \text{or, } \div 2 \\
 -82 + 527
 \end{array}$$

Consequently the functions are

$$X = x^4 - 14x^3 + 66x^2 - 123x + 90$$

$$X_1 = 4x^3 - 42x^2 + 132x - 123$$

$$X_2 = 10x^2 - 62x + 47$$

$$X_3 = -82x + 527$$

$$\therefore X_4 = -$$

.

As there is one variation in the signs of the leading terms we infer that the equation has a pair of imaginary roots. Also since there is one variation for $x = \infty$, and three for $x = 0$, we conclude that the two real roots are both positive. By substituting $0, 1, 2, \dots$ for x we find that X vanishes for $x = 5$, and for $x = 6$; these therefore are the two real roots of the equation.

(98.) The following equation is one of some celebrity. It is that to which Dr. WALLIS was led in his solution of "COLONEL TITUS's Problem." One of the roots of this equation is determined by Mr. HORNER's method, as far as fifteen places of decimals, in an historical dissertation on the problem of Involution, published in the *Companion to the Almanack* for 1839;* and the same root is computed to seventeen places of decimals, by the NEWTONIAN

* Written, we suspect, by Professor DE MORGAN, who, in reference to the equation mentioned, says, "This is the equation resulting from the celebrated problem proposed by Dr. PELL to Colonel SILAS TITUS, and by him to Dr. WALLIS, who seems to have considered it as a sort of challenge from PELL, and has discussed it in a most masterly manner. A great deal may be found on the subject of this very equation in the *Scriptores Logarithmici*, and other publications, of Baron MASENER."

To this we may add that the problem itself is to find x , y , and z , from the three equations,

$$x^2 + yz = 16, \quad y^2 + xz = 17, \quad z^2 + xy = 18.$$

It has been solved by many algebraists; as WALLIS, HALLEY, IVORY, and FRENCH; but one of the earliest solutions that the author of this work has met with was given in 1825, by Mr. JOHN WHITLEY, of Bradford, a geometer of great ability, in a periodical work published at Liverpool, under the title of *The Apollonius*, and conducted by another excellent geometrician, Mr. J. H. SWALE.

Professor DAVIES too has discussed this celebrated problem at length, in his recently published *Key to Hutton's Course of Mathematics*, a work replete with curious and useful information, and forming a very valuable acquisition to the library of the mathematical student.

method of approximation, by Mr. LOCKHART.* The equation is this:

$$7. \quad x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0,$$

and the numerical operation for the discovery of the number and situations of its roots is as follows:

$$\begin{array}{r}
 4+240 \quad -3996 \quad +14937 \\
 \times \\
 1-80 \quad +1998 \quad -14937 \quad +5000 \\
 \hline
 -20 \quad -4800+79920-298740 \\
 1 \quad -3996+14937 \\
 4 \quad 7992-59748 \quad +20000 \\
 \hline
 804-35109+278740 \\
 \hline
 804+35109-278740 \\
 \times \\
 4-240 \quad +3996 \quad -14937 \\
 \hline
 -4377 \quad -153672093+1220044980 \\
 268 \quad -74702320 \\
 804 \times 67 \quad 215256528 \quad -804626316 \\
 \hline
 13117885 \quad -415418664.
 \end{array}$$

Consequently the functions are

$$X = x^4 - 80x^3 + 1998x^2 - 14937x + 5000$$

$$X_1 = 4x^3 - 240x^2 + 3996x - 14937$$

$$X_2 = 804x^2 - 35109x + 278740$$

$$X_3 = 13117885x - 415418664$$

$$X_4 = +$$

* We cannot refer to this venerable mathematician without expressing our admiration of the ardour and perseverance that, at the age of eighty years, supports him through the long and laborious calculations in which he is engaged. Our readers will be both gratified and instructed by the perusal of the following works recently published by that gentleman:

1. "*Resolution of Equations*," quarto, Oxford, 1837.
2. "*Resolution of two Equations*," &c. 1839.
3. "*Extension of the Celebrated Theorem of C. STURM*," 1839.

Hence the roots are all real; they are moreover all positive, since for $x = 0$ there are four variations. The situations of these roots are found thus:

	X	X ₁	X ₂	X ₃	X ₄		
for $x = 0$		+	-	+	-	+ <i>four variations</i>	
	= 1		-	-	+	- + <i>three</i>	
	= 10		-	+	+	- +	
	= 20		+	+	-	- + <i>two</i>	
	= 30		+	-	-	- +	
	= 40		+	+	+	+	<i>no variation.</i>

We conclude therefore that there is one root between 0 and 1, one between 10 and 20, and two between 30 and 40; that is, the leading integers of the roots are *zero*, *ten*, and *thirty*, the latter number belonging to a pair of roots.

(99.) The work of the foregoing example is well adapted to show the amount of numerical labour usually involved in the application of STURM'S theorem to a complete equation of the fourth degree, whose coefficients are rather high numbers. Means will be furnished hereafter for greatly reducing this labour, which is, to a considerable extent, superfluous. But, as before stated, we think it desirable to postpone all abbreviating contrivances for the present, in order that ample illustration of the general mode of derivation may first be exhibited to the student.

8. Required the number and situations of the real roots of the equation

$$x^5 - 36x^3 + 72x^2 - 37x + 72 = 0$$

The leading polynomials here are

$$x^5 + 0x^4 - 36x^3 + 72x^2 - 37x + 72,$$

and

$$5x^4 + 0x^3 - 108x^2 + 144x - 37,$$

and the work is as follows :

$$\begin{array}{r}
 5+0+108-144 \quad +37 \\
 \times \\
 1+0 \quad -36 \quad +72 \quad -37+72 \\
 \hline
 0 \left[\begin{array}{l} 108-144 \quad +37 \\ 5-180+360-185+360 \end{array} \right. \\
 \hline
 72-216+148-360 \\
 \text{or, } \div 4 \\
 18-54 \quad +37-90 \\
 \hline
 18+54 \quad -37 \quad +90 \\
 \times \\
 5+0 \quad -108 \quad +144-37 \\
 \hline
 15 \left[\begin{array}{l} 810 \quad -555+1350 \\ 5 \quad -185 \quad +450 \\ 18-1944+2592 \quad -666 \end{array} \right. \\
 \hline
 1319-2487 \quad -684 \\
 \hline
 1319+2487 \quad +684 \\
 \times \\
 18-54 \quad +37-90 \\
 \hline
 -490 \times 54^* \left[\begin{array}{l} -65806020 \quad -18098640 \\ 1319 \times 18 \quad 16239528 \\ 1319^2 \quad 64371157-156578490 \end{array} \right. \\
 \hline
 -14804665+174677130 \\
 \text{or, } \div 5 \\
 -2960933 \quad +34935426
 \end{array}$$

* As the number 2487 is divisible by 3, and 3 times 18 make 54, the leading multiplier in the side column may be considered as formed from

$$\begin{array}{r}
 1319+829 \\
 \times \\
 54-54
 \end{array}$$

that is, from

$$(829-1319).54 = -490 \times 54.$$

consequently, STURM's functions are

$$X = x^5 - 36x^3 + 72x^2 - 37x + 72$$

$$X_1 = 5x^4 - 108x^2 + 144x - 37$$

$$X_2 = 18x^3 - 54x^2 + 37x - 90$$

$$X_3 = 1319x^2 - 2487x - 684$$

$$X_4 = -2960933x + 34935426$$

$$X_5 = -$$

Substituting $-\infty$ and $+\infty$ successively for x in these functions, we have the series

$$x = -\infty, \dots - + - + + - \dots \text{four variations}$$

$$x = +\infty, \dots + + + + - - \dots \text{one variation.}$$

Hence the equation has only *three* real roots.

To determine the situations of these roots, we have

$$\text{for } x = 0 \dots + - - - + -$$

$$= 1 \dots + + - - + -$$

$$= 2 \dots + - - - + -$$

$$= 3 \dots - - + + + -$$

$$= 4 \dots - + + + + -$$

$$= 5 \dots +$$

As there are two more variations for $x=0$ than for $x = +\infty$,

we infer that there are two positive roots; the third root therefore must be negative, because the sign of the last term of X is positive. The variations show that these positive roots are the one between 2 and 3, and the other between 4 and 5. By substituting the negative numbers $-1, -2, -3$, &c. in the single function X , or the positive numbers 1, 2, 3, &c. after changing the alternate signs in that function, we find that the remaining root lies between -6 and -7 .

In order to diminish the number of substitutions, and to discover at once the immediately superior limit to the sought root, which, when taken positively, is the greatest root of the equation

$$x^5 - 36x^3 - 72x^2 - 37x - 72 = 0,$$

we have only to seek the smallest value of x , that will satisfy the inequality

$$x^5 > 36x^3 + 72x^2 + 37x + 72,$$

or

$$1 > \frac{36}{x^2} + \frac{72}{x^3} + \frac{37}{x^4} + \frac{72}{x^5}.$$

This value we immediately see must be 7,* so that the greatest root must lie between 6 and 7, and therefore the root of the original equation between -6 and -7 .

The foregoing example is given by BOURDON,† who has

* However, 6 is more immediately seen to be a close *inferior* limit, so that we might commence our substitutions with 6.

† *Algèbre*, p. 542, 1837.

deduced incorrect values for X_3 and X_4 , and has moreover assigned a wrong place to one of the positive roots.

(100.) We shall now show the application of the theorem to literal equations of the fourth and fifth degrees.

9. Required STURM's functions for the equation

$$x^4 + ax^3 + bx + c = 0$$

$$\begin{array}{r}
 4+0-2a-b \\
 \times \\
 1+0+a+b+c \\
 \hline
 0 \left[\begin{array}{l} -2a-b \\ 4a+4b+4c \end{array} \right. \\
 \hline
 -2a-3b-4c \\
 \hline
 -2a+3b+4c \\
 \times \\
 4+0+2a+b \\
 \hline
 3b \left[\begin{array}{l} 9b^2-12bc \\ -2a-8ac \end{array} \right. \\
 a^2 \left[\begin{array}{l} 2a^2+a^2b \\ A+B \end{array} \right. \\
 \hline
 A-B \\
 \times \\
 -2a-3b-4c \\
 \hline
 2aB-3bA \left[\begin{array}{l} -2aB^2+3bAB \\ A^2-4cA^2 \end{array} \right. \\
 \hline
 4cA^2-3bAB+2aB^2.*
 \end{array}$$

* This final result, as before remarked, is the same as that which would arise from putting $-\frac{B}{A}$ for x in the result preceding that whence this expression for x is deduced; that is, in $-2ax^2-3bx-4c$, and then changing the signs.

Hence the functions are

$$X = x^4 + ax^3 + bx + c$$

$$X_1 = 4x^3 + 2ax + b$$

$$X_2 = -2ax^2 - 3bx - 4c$$

$$X_3 = (8ac - 2a^2 - 9b^2)x - b(a^2 + 12c)$$

$$= Ax + B$$

$$X_4 = 4cA^2 - 3bAB + 2aB^2.$$

If $X_4 = 0$, the equation will have a pair of equal roots, namely, the roots of the equation

$$(Ax + B)^2 = 0,$$

each root being $= -\frac{B}{A}$.

The equation containing the other two roots is therefore

$$\frac{X}{(Ax + B)^2} = 0, \text{ or } x^2 - 2\frac{B}{A}x + \frac{A^2}{B^2}c = 0.$$

If $X_3 = 0$, then there will be two pair of equal roots in the proposed, and the equation

$$2ax^2 + 3bx + 4c = 0$$

will furnish one of each pair, unless it should so happen that this latter equation has itself equal roots; that is, unless $9b^2 = 32ac$, in which case the proposed equation will have three equal roots, of which the equation $X_2 = 0$ contains two. These two being together equal to the coefficient of x in X_2 , when the sign of that coefficient is changed, and the division by $2a$, the coefficient x^2 performed, it follows that the expression for each of the three equal roots is $x = -\frac{3b}{4a}$; and as in the proposed equation the sum of all the roots, with changed signs, is 0, it follows that the fourth root is $\frac{9b}{4a}$.

(101.) Given the equation

$$10. \quad x^4 + ax^3 + bx^2 + cx + d = 0$$

to determine STURM'S functions.

The two leading polynomials are in this case

$$x^4 + 0x^3 + ax^2 + bx + c,$$

and

$$5x^4 + 0x^3 + 3ax^2 + 2bx + c,$$

and the work of derivation is as follows:

$$\begin{array}{r} 5+0-3a-2b-c \\ \times \\ 1+0+a+b+c+d \end{array}$$

$$\begin{array}{r|l} 0 & \\ 1 & -3a-2b-c \\ 5 & 5a+5b+5c+5d \\ \hline & -2a-3b-4c-5d \end{array}$$

$$\begin{array}{r} -2a+3b+4c+5d \\ \times \\ 5+0+3a+2b+c \end{array}$$

$$\begin{array}{r|l} 15b & 45b^2+60bc+75bd \\ -10a & -40ac-50ad \\ 4a & 12a^2+8a^2b+4a^2c \end{array}$$

$$\underline{A + B + C}$$

$$A-B-C$$

$$\begin{array}{r} \times \\ -2a-3b-4c-5d \end{array}$$

$$\begin{array}{r|l} 2aB-3bA & -(2aB-3bA)B-(2aB-3bA)C \\ -2aA & 2aAC \\ A^2 & 4cA^2-5dA^2 \end{array}$$

$$\underline{D + E}$$

$$D-E$$

$$\begin{array}{r} \times \\ A+B+C \end{array}$$

$$BD-AE \quad \begin{array}{r|l} & -(BD-AE)E \end{array}$$

$$\begin{array}{r|l} D^2 & CD^2 \end{array}$$

$$\underline{F.}$$

Hence STURM's functions are as follow:

$$X = x^5 + ax^3 + bx^2 + cx + d$$

$$X_1 = 5x^4 + 3ax^2 + 2bx + c$$

$$X_2 = -2ax^2 - 3bx^2 - 4cx - 5d$$

$$X_3 = Ax^2 + Bx + C$$

$$X_4 = Dx + E$$

$$X_5 = F;$$

where A, B, C, D, E, F, are the following functions of the coefficients, namely,

$$A = 40ac - 12a^3 - 45b^2$$

$$B = 50ad - 8a^2b - 60bc$$

$$C = -4a^2c - 75bd$$

$$D = 4cA^2 + (2aB - 3bA)B - 2aAC$$

$$E = 5dA^2 + (2aB - 3bA)C$$

$$F = (BD - AE)E - CD^2.$$

* If by means of these forms we can discover, without actual calculation, the signs of A and D, and are able also, from an estimate of the values of the preceding coefficients, to predict the sign of F, we shall obviously be in possession of information sufficient to enable us to find at once the number of real roots in the equation. The character of these roots, as to positive and negative, will also be ascertained from a comparison of the signs of C, E, and F, in connexion with the sign of *d*. But for a very complete and detailed examination of all these circumstances, the student is referred to an interesting paper by Professor GILL, of New York, published in the eighth number of *The Mathematical Miscellany*, an American periodical very ably conducted, and well worthy of attention from students of the modern analysis. The work is published in London by Groombridge, Paternoster Row, at intervals of six months.

An abridgment of Mr. GILL's paper, here adverted to, may be seen in the *Ladies' Diary* for 1839.

The examination referred to will place in a striking light the superiority of

(102.) In applying these general formulas to any particular example, there will be no occasion to execute the actual computation of F , as we have already remarked, since the sign of F , which is all we require to know about it, is easily inferrible from X_1 and X_3 . It is easy to see, however, that there will be but little saving of actual labour effected by employing these forms in preference to going through the process which leads to them, at least as respects the more complicated functions X_3 and X_4 . The functions preceding these, may indeed sometimes be obtained more readily by the formulas, or rather by these somewhat modified. Thus, for the purpose of computation, the following will usually be found to be the more convenient method of writing the expressions for A , B , and C , each of the former expressions for these quantities being divided by a^2 .

$$A = 40 \frac{c}{a} - 12a - 45 \frac{b}{a} \cdot \frac{b}{a}$$

$$B = 50 \frac{d}{a} - 8b - 60 \frac{b}{a} \cdot \frac{c}{a}$$

$$C = -4c - 75 \frac{b}{a} \cdot \frac{d}{a}$$

STURM's method over that of LAGRANGE, who was fully aware however of the labour involved in his own process; and accordingly remarks with great candour, after discussing the general equation of the *fourth* degree, "On pourrait de même trouver les conditions qui rendent les racines des équations du cinquième degré toutes réelles, ou en partie réelles et en partie imaginaires; mais comme, dans ce cas, l'équation des différences monterait au degré $\frac{5 \cdot 4}{2} = 10$, le calcul deviendrait extrêmement prolix et embarrassant." *Traité de la Résolution des Equations Numériques*, p. 43, 1828.

We need scarcely add to this quotation, that, notwithstanding the recent improvements in this subject, the profound and elegant work of LAGRANGE will richly repay the analytical student for a careful perusal.—See NOTE at the end.

(103.) As an application of these forms, let us take the equation of the fifth degree before given, namely,

$$x^5 - 36x^3 + 72x^2 - 37x + 72 = 0.$$

Then, from the general expression for X_2 , we have immediately

$$X_2 = 72x^3 - 216x^2 + 148x - 360$$

$$\text{or, } \div 4$$

$$18x^3 - 54x^2 + 37x - 90.$$

Also, since

$$\frac{b}{a} = -2, \quad \frac{c}{a} = 1\frac{1}{3}, \quad \frac{d}{a} = -2,$$

we have, from the general expressions above,

$$A = 41\frac{1}{3} + 432 - 180 = 293\frac{1}{3}$$

$$B = -100 - 576 + 123\frac{1}{3} = -552\frac{2}{3}$$

$$C = 148 - 300 = -152$$

Consequently,

$$X_3 = 293\frac{1}{3}x^2 - 552\frac{2}{3}x - 152;$$

or, without fractions,

$$X_3 = 1319x^2 - 2487x - 684.$$

A glance at the general expressions for D and E will suffice to show, that the very same labour is involved in their

computation, as is necessary to complete the step for X_4 by the regular process: we should therefore recommend X_4 to be derived in this latter way.

(104). We should find, by proceeding in a similar manner with a general equation of the sixth degree, wanting its second term; that is, with the equation

$$11. \quad x^6 + ax^4 + bx^3 + cx^2 + dx + e = 0$$

that

$$X_2 = -2ax^4 - 3bx^3 - 4cx^2 - 5dx - 6e,$$

and that the coefficients A, B, C, D, of the next function

$$X_3 = Ax^5 + Bx^4 + Cx + D$$

are as follow:

$$A = 24ac - 8a^3 - 27b^2$$

$$B = 30ad - 6a^2b - 36bc$$

$$C = 36ae - 4a^2c - 45bd$$

$$D = -2a^2d - 54be$$

or somewhat more commodiously,

$$A = 4.6 \frac{c}{a} - 8a - 3.9 \frac{b}{a} \cdot \frac{b}{a}$$

$$B = 5.6 \frac{d}{a} - 6b - 4.9 \frac{b}{a} \cdot \frac{c}{a}$$

$$C = 6.6 \frac{e}{a} - 4c - 5.9 \frac{b}{a} \cdot \frac{d}{a}$$

$$D = -2d - 6.9 \frac{b}{a} \cdot \frac{e}{a}$$

And we might, with the same ease, obtain the functions to the same extent for any higher equation. There would indeed be but little difficulty in extending the series further, so as to complete the set. But the algebraical expressions for the coefficients in the subsequent functions would offer no facilities for the actual computation of them, beyond those of which we might avail ourselves in the derivation of them by the regular process ; so that it would be useless to exhibit them in their algebraic form.

(105.) It must be acknowledged, however, that it is always in the derivation of these subsequent functions that the irksomeness of the numerical labour is most felt, because of the rapid rate at which the coefficients generally increase as we approach towards the final step of the operation.

It is true that these large numbers may be sometimes reduced by the suppression of common factors ; but it is not always easy to discover these, even when they exist, except indeed they are of such small amount as to render the division by them, in the case of these very large numbers, of little or no importance.

It is plain, therefore, that any considerable reduction of the work here, provided it could always be made without shaking our confidence in the numerical results, would constitute a capital improvement in STURM'S THEOREM.

But if the numerical coefficients which enter these advanced functions are always required to be exhibited in full, with the cumbersome redundancy of figures which they so frequently present, then, we confess, that we have but little expectation of any reduction worth notice ever being made in the work by which these large numbers are obtained. The case would be different if the labour complained of consisted in the intricacy or complication of the means by which the several results are arrived at : we might then reasonably hope that, as in other practical en-

quiries, time and application would eventually bring to light simpler and shorter methods of attaining the same end. But here there is no complication of means; each step of the work may almost be said to involve but the fewest possible operations, and these of the simplest possible character. The difficulty felt in practice is attributable solely to the magnitude of the numbers with which we have to deal; and we cannot conceive of any reduction of this difficulty being practicable, that has not for its main object the contracting and confining within moderate bounds these overgrown coefficients.

(106.) We are disposed to think that the general mode of conducting the numerical work of STURM's theorem, as exhibited in the preceding pages of this dissertation, reduces the operation to the utmost degree of practical convenience. To give it perfection there only remains to be united with it, in its closing steps, some abbreviating contrivance, of easy application, that shall at once diminish the actual labour, and restrain the resulting numbers within moderate limits.

As to any means being hereafter discovered for lessening the labour of the common multiplication of large numbers, thus increasing the facilities at present furnished by our tables of logarithms and of squares and products, we have, as already remarked, little or no hope.* We think such a reduction of labour to be altogether impracticable, by any purely numerical or algebraical contrivance whatever. It may possibly be accomplished by instrumental or mechanical methods; and we should suppose that the several steps of STURM's theorem, in their unabbreviated form, however large the numbers involved in them, might be

* BARLOW's Tables will be found of much assistance in the practical operations of STURM's Theorem. The work has been recently reprinted by the Society for the Diffusion of Useful Knowledge.

worked out, with comparative ease, by the aid of the admirable *calculating engine* of Mr. BABBAGE.

But even admitting the necessity of undertaking this labour by the common arithmetical process, which necessity, however, as we shall shortly show, does not exist, yet the amount of work in STURM's theorem has, we think, been somewhat overrated in the comparisons that have been made between it and other modes of operation. When the *labour* is made the ground of rejection of this method in favour of any other at present known, it is surely overlooked, that be this labour whatever it may, yet it is indisputable that STURM's is not only the *simplest*, but is also the *shortest* unfailing method that has ever been proposed. All the work which STURM's method requires is essentially comprehended in the method of LAGRANGE, in conjunction with the additional labour peculiar to his process. The same is true of that of FOURIER or BUDAN. And no *infallible* rule for finding the number and situations of the real roots of a numerical equation, chosen at random, can be adduced, in the present state of analysis, that shall not imply the process for finding the common measure of the proposed polynomial and its first derived function.* What entitles STURM's method to the precedence of them all is, that although it also requires the finding of this common measure, yet it requires nothing else: this is the single and unique operation that comprises the whole of his process, and it is certainly an operation of a very elementary character.

(107.) Thus the labour of STURM's method, even when viewed in the most unfavorable light; that is, when all

* "Avant d'entreprendre la résolution d'une équation par quelque méthode que ce soit, il est toujours nécessaire de s'assurer si elle a des racines égales." LAGRANGE, *Résolution des Equations Numériques*, p. 115, ed. 1826.

It is almost superfluous to add that the search after equal roots necessarily involves the operation for the common measure adverted to in the text.

abbreviations of it are precluded, is, after all, a labour so entirely free from ambiguity and intricacy, a labour of so ordinary a kind, that the merest tyro is competent to perform it. Every successive step is characterized by the same simplicity and undeviating uniformity, each being the model or pattern of the next in succession, so that the work can be carried on with little or no mental exertion, and can be broken off at any step, and resumed again at pleasure without endangering the accuracy of the operation.

(108.) In certain cases the transformed equations of BUDAN will, no doubt, effectually make known the number and nature of the roots, with, comparatively, a trifling amount of labour; such will happen when these roots are all real, and moderately wide apart. It is possible, in some practical enquiries, that the problem which conducts to our equation may furnish extraneous information as to the nature of the roots, or may afford strong grounds for conjecturing that the roots can none of them be imaginary, nor yet nearly equal. Under such circumstances it may be prudent to apply the tests of BUDAN. But when, as usually happens, we are in total ignorance of the constitution of our equation, nothing can effectually supply the place of STURM's infallible method.

Nevertheless, either from prejudice, or from an imperfect apprehension of the comparative merits of the two rules, that of FOURIER is sometimes represented as superior to the method of STURM. It would be easy to show, from the examples selected by FOURIER himself for the purpose of illustrating his own rule, that in cases of any delicacy STURM's process surpasses his, as well in ease and simplicity, as in certainty. In confirmation of this we shall take, from FOURIER's book, an equation of this kind, and one in which every figure employed in the work is actually exhibited in the several steps, there being no bye operations whatever. By a comparison of the two pro-

cesses it will be seen that that here given is by far the simpler.

12. Required the number and situations of the real roots of the equation

$$x^4 - 4x^3 - 3x + 23 = 0^*$$

$$\begin{array}{r}
 4+12+0+3 \\
 \times \\
 1-4+0-3+23 \\
 \hline
 -1 \mid -12-0-3 \\
 1 \mid \quad 0+3 \\
 4 \mid \quad 0-12+92 \\
 \hline
 \quad 12+9-89 \\
 \hline
 \quad 12-9+89 \\
 \times \\
 \quad 4-12-0-3 \\
 \hline
 -15 \mid 135-1335 \\
 4 \mid 356 \\
 12 \mid \quad 0 \quad -36 \\
 \hline
 -491+1371
 \end{array}$$

* FOURIER, *Analyse des Equations*, p. 137.

As this equation, when put in a complete form, is

$$x^4 - 4x^3 \pm 0x^2 - 3x + 23 = 0,$$

it is evident at once, from the criterion of DE Gua, that two of its roots are imaginary, (see *Theory of Equations*, p. 61.) To remove all doubt as to the nature of the other roots, FOURIER recommends, in addition to his own peculiar method of examination, that we should go through the very operation which constitutes the whole of STURM's process: the operation, namely, of seeking the common measure of the proposed function and its derivative.

Writing now the functions of STURM we have

$$X = x^4 - 4x^3 - 3x + 23$$

$$X_1 = 4x^3 - 12x^2 - 3$$

$$X_2 = 12x^2 + 9x - 89$$

$$X_3 = -491x + 1371$$

$$\therefore X_4 = -$$

from which we see that there are two real and positive roots, inasmuch as two variations are lost between $x = 0$ and $x = \infty$. Moreover, since the greatest negative coefficient of the proposed equation increased by unity is 5, it follows that 5 is a superior limit to these roots. But since for such limit

$$x^4 + 23 > 4x^3 + 3x$$

or

$$1 + \frac{23}{x^4} > \frac{4}{x} + \frac{3}{x^3}$$

it is easily seen that 4 is also a superior limit. The roots are thus readily found to lie between 2 and 3, and between 3 and 4.

But a more striking example of the great inferiority of FOURIER's mode of proceeding, in any but the simplest cases, is furnished by the equation

$$13. \quad x^5 + x^4 + x^3 - 2x^2 + 2x - 1 = 0$$

the analysis of which, as detailed in the work before referred to, (*Anal. des Eq.* p. 146,) is very laborious; requiring, in conjunction with several trial-calculations, no

fewer than *three* operations for the common measure. The whole process, by the theorem of STURM, is as follows:

$$\begin{array}{r}
 5-4-3+4-2 \\
 \times \\
 1+1 \quad +1 \quad -2+2-1 \\
 \hline
 1 \quad -4 \quad -3 \quad +4-2 \\
 5 \quad -15+20-10 \\
 25 \quad 25-50+50-25 \\
 \hline
 \quad \quad -6+33-44+27 \\
 \hline
 \quad \quad -6-33+44-27 \\
 \times \\
 \quad \quad 5+4 \quad +3 \quad -4+2 \\
 \hline
 -63 \quad 2079-2772+1700 \\
 -10 \quad -440 \quad +270 \\
 12 \quad 36 \quad -48 \quad +24 \\
 \hline
 \quad \quad -1675+2550-1724
 \end{array}$$

Now, since it is immediately seen that

$$4 \times 1675 \times 1724 > 2550^2,$$

the operation terminates at this step.* Moreover, as two

* The abrupt conclusion of the work, as soon as we arrive at a function, the roots of which are known to be imaginary, is an important feature in STURM's method, and is often the means of saving a considerable portion of the numerical labour. There is nothing analogous to it in the method of FOURIER.

The criterion of imaginary roots is of very easy application in equations of the second degree. It would be easy too to investigate an unfailing a criterion for those of the fourth degree; and, if such criterion were of convenient application, it would be worth while to do so. But the complicated character of the test is such as to render it quite useless in connexion with STURM's theorem. We have given it however in Note I. at the end, where it is shown that one of the conditions furnished by STURM, for the reality of the roots, is unnecessarily complicated; and, what is a little remarkable, the tests of LAGRANGE, generally of such laborious application, suggest, in this case, the proper simplification of STURM's criteria.

variations are furnished by the leading signs for $x = -\infty$, and only one for $x = +\infty$, we infer that there is but one real root.

From the sign of the last term in the proposed equation, we conclude that the real root must be positive, and it is immediately seen to lie between 0 and 1.

If an example be taken in which there are not only imaginary roots, but also pairs of roots very close together, the superiority of STURM's method will be still more conspicuous. FOURIER, however, has but sparingly introduced into his work examples of this kind.

(109.) We shall now endeavour to obviate the objection advanced against STURM's method on the ground of the intolerable labour of executing the closing steps of the work in equations beyond the fourth degree, and of the great extent to which the coefficients in the resulting functions reach.

We propose to put an effectual check upon this tendency of the successive coefficients to increase to an unmanageable extent; this check is suggested by the following considerations:

There are three multipliers connected with every step of the work; these, in our arrangement of the process, occupy the side column in each step; and we have seen how to reduce their magnitude whenever they involve a factor common to all three. But to the second and third there is *always* a common factor; and one too of considerable magnitude in the advanced steps. It is plain that this factor may always be suppressed, provided the first multiplier be divided by it, and the quotient be put to supply the place of this multiplier. Even taking into account the trouble of performing this division, as indeed we ought, there will still be a considerable saving of labour, when the second and third multipliers are very large, by this simple modification; but as we wish to reduce the results of these

steps, as well as the labour of executing them, we shall perform the several multiplications required, not at full length, but according to the contracted method, so commonly resorted to in multiplication of decimals, when we desire accuracy in our results only to a certain proposed extent of leading figures. By introducing this means of abbreviation into the work of STURM's steps, we may restrict the several resulting coefficients to any number of leading figures we please, which shall be all correct as far as they extend, and the labour will thus be saved, as well as the inconvenience avoided, of introducing into the closing steps of the work a long array of redundant figures.

A glance at the last step of the operation at page 162. will be enough to show that the amount of labour expended on it, and the extent of figures obtained in the result, far exceed the actual demands of the case; two or three only of the leading figures — 296....., 349....., would have been amply sufficient; and to obtain these, by the means described above, would have cost us comparatively but very little trouble.

(110.) But, instead of returning upon this step for the purpose of illustrating the importance of the abbreviations here recommended, it will be more instructive to take an example of much greater difficulty; and we therefore select the following complete equation of the fifth degree.

14. Required the number and situations of the real roots of the equation

$$x^5 + 7x^4 - 144x^3 + 611x^2 - 928x + 362 = 0.$$

The leading functions here are

$$X = x^5 + 7x^4 - 144x^3 + 611x^2 - 928x + 362,$$

and

$$X_1 = 5x^4 + 28x^3 - 432x^2 + 1222x - 928.$$

and the entire work by the general method, abbreviated after the first step, in accordance with the foregoing directions, is as follows :

$$\begin{array}{r}
 \begin{array}{r}
 5-28 \quad +432 \quad -1222 \quad +928 \\
 \times \\
 1+7 \quad -144 \quad +611 \quad -928+362 \\
 \hline
 7 \quad -196 \quad +3024 \quad -8554 \quad +6496 \\
 5 \quad 2160 \quad -6110 \quad +4640 \\
 25 \quad -3600 \quad +15275 \quad -23200 \quad +9050 \\
 \hline
 1636-12189+27114-15546 \\
 \hline
 1636+12189-27114+15546 \\
 \times \\
 5+28 \quad -432 \quad +1222-928 \\
 65\cdot2524 \quad 795362-1769254+1014414 \\
 5 \quad -135570 \quad +77730 \\
 1636 \quad -706752+1999192-1518208 \\
 \hline
 46960 \quad -307668 \quad +503794 \\
 \hline
 46960+307668-503794 \\
 \times \\
 1636-12189+27114-15546 \\
 -1470\cdot4 \quad -4524+7408 \\
 1636 \quad -8242 \\
 46960 \quad 12733-7300 \\
 \hline
 33 \quad -108.
 \end{array}
 \end{array}$$

It may be proper to remark, for the better elucidation of the foregoing work, that the leading multiplier $65\cdot2524$, in the second step, is extended only to a sufficient number of decimals to secure accuracy in the *integral parts* of the several products resulting from it. These products are all obtained by contracted multiplication, the decimal part being rejected.

In the next following step of the work it is proposed to

secure accuracy only as far as the first *four* or *five* figures in the several products; in other words, *five* figures are uniformly rejected from the right, in each product, or rather these products, wanting the five terminating figures, are obtained by contracted multiplication. As in the preceding step so here, the leading multiplier — 1470·4 contains none of those decimals whose influence would be confined only to the figures thus rejected.

We may mention, in reference to the manner in which these leading multipliers are obtained, that the operations implied in

$$\begin{array}{r} 1636 + 12189 \\ \times \\ 5 + 28 \end{array}$$

and in

$$\begin{array}{r} 46960 + 307668 \\ \times \\ 1636 - 12189 \end{array}$$

are not actually performed, and the division in the one case by 1636, and in the other case by 46960, afterwards executed; because this course of proceeding would involve unnecessary trouble.

But the first of these multipliers is obtained by dividing 12189 by 1636, then multiplying the quotient 7·45048 by 5, and adding 28 to the product. In like manner, to obtain the other multiplier, 307668 is divided by 46960, the quotient 6·5517 is then multiplied by 1636, and — 12189 added to the product.

(111.) The above example is by no means a favorable one for our abbreviated method; as three roots of the proposed equation lie pretty closely together. We have a sufficiently plain intimation of this in the circumstance that the final function turns out to be comparatively near to zero; and, in addition to this, the results of the second

step are such that, considering the magnitudes of the numbers, four times the product of the extreme terms, differs but little from the square of the mean term : the less this difference is, the more nearly of course do the roots of the quadratic, of which these numbers are the coefficients, approach to equality.

It is quite clear, however, that the final function, just adverted to, and the sign of which we infer as usual from substituting $\frac{108}{33}$ for x in the preceding quadratic function, is of finite magnitude ; and, with change of sign, is *plus*. If in any case this final function, or indeed any preceding function, should be actually zero, most likely there will be equal roots ; yet in case the roots, instead of being rigorously equal, should have a minute difference, attention should be paid to the accuracy of the terminating right-hand figure of each of the coefficients of the function at which our work terminates.

In the example above, although as just remarked one of some delicacy, it is of no moment whether or not the 33 and the 108 err by a unit in the last figures 3 and 8. Nevertheless, in the multiplications which contribute to produce these numbers, it is as well to notice, that the 33 is in reality 33·3 ; and the two numbers over the 108 are, in strictness, 7300·4 and 7407·8 ; so that 108 is a unit in excess. The correct value is therefore 107.

We have just observed that when our final result is zero, *most likely* there will be equal roots. We cannot affirm that in such a case there are *necessarily* equal roots, because our result, for aught we know to the contrary, may be zero only within the limits to which the coefficients last deduced are restricted ; and thus the roots may be equal only as far as this prescribed extent. But for a figure or two, we may always in such a case safely infer the equality of a pair of roots. The situations of the roots, in such critical instances, will therefore always be clearly

pointed out; the leading figure of each, and generally two or three of the leading figures, being made known by equating our terminating function, as for instance, the function of the first degree above, to zero.

It appears then we can never be at a loss, in consequence of our abridged mode of working, as to the character of the roots of the proposed equation; although, when roots have several leading figures the same in each, we may not be able to say whether they differ at a remote figure or not without additional examination.* But we shall revert to this presently.

(112.) We shall now proceed to determine the character and situations of the roots of the preceding equation. The functions just determined are

$$X = x^5 + 7x^4 - 144x^3 + 611x^2 - 928x + 362$$

$$X_1 = 5x^4 + 28x^3 - 432x^2 + 1222x - 928$$

$$X_2 = 1636x^3 - 12189x^2 + 27114x - 15546$$

$$X_3 = 46960x^2 - 307668x + 503794$$

$$X_4 = 33x - 107$$

$$X_5 = +$$

As the signs of the *leading* terms of these functions are all *plus*, we infer that the roots are all real. And as the signs of the *final* terms present four variations, we conclude

* It will be proved in Note II. at the end, that when the last, or constant function is actually zero, then the preceding function, or that of the first degree in x , when superfluous factors are expunged, must have its leading coefficient a factor of that in the proposed equation; and the remaining number also a factor of the final number in the proposed equation. It will always be easy to ascertain whether or not such is the case.

that there are four positive roots.* We shall seek the places of these,

$$\begin{aligned}
 \text{for } x = 0 & \dots + - - + - + \text{ four variations} \\
 & = 1 \dots - - + + - + \text{ three } ,, \\
 & - 2 \dots - + + + - + \\
 & = 3 \dots - + + + - + \\
 & = 4 \dots + + + + + + \text{ no variation.}
 \end{aligned}$$

Hence there is one positive root between 0 and 1, and there are three between 3 and 4.

The negative root is found, from the single polynomial X , to lie between 10 and 20; that is, its first figure is 1 in the tens' place.

The simplifications introduced into the foregoing work, by means of the abbreviations made use of in the second and third steps, are very considerable; as will be soon felt by any one who will take the trouble of going through the several multiplications in their unabbreviated form.

Mr. LOCKHART, in the publication on STURM's *Theorem*,

* The number of positive and negative roots in any equation may always be readily ascertained from an inspection merely of the leading and final signs in STURM's functions, without making any actual substitutions in them. The positive roots will always be equal in number to the excess of the variations in the final signs above the variations in the leading signs.

This rule for the number of positive roots will not require any modification even when the functions X, X_1, X_2 , &c. are fewer in number than $n + 1$ in an equation of the n th degree. But as regards the imaginary roots there will always be as many pairs of them as there are functions wanting to make up the ordinary number $n + 1$, besides the pairs indicated by the variations in the leading signs of the incomplete series. We shall have occasion shortly to notice a modification necessary to be introduced into the numerical operation of STURM's theorem whenever these incomplete series occur.

before referred to, has executed this labour, availing himself of only the ordinary abridgments; and he finds the complete functions to be as follow :

$$X = x^5 + 7x^4 - 144x^3 + 611x^2 - 928x + 362$$

$$X_1 = 5x^4 + 28x^3 - 432x^2 + 1222x - 928$$

$$X_2 = 1636x^3 - 12189x^2 + 27114x - 15546$$

$$X_3 = 3073059x^2 - 20133742x + 32968246$$

$$X_4 = 20969919426169x - 67576648427974$$

$$X_5 = +$$

The three roots lying between 3 and 4 are

$$3.213....., 3.229....., 3.414.....$$

It may be interesting to notice, as corroborative of the accuracy of both Mr. LOCKHART's calculations and mine, that if the number 209699..... be divided by 33.3, the quotient 6297....., taken as a divisor of 675766....., gives exactly 107.

(113.) We have already adverted to the circumstance that the remainder X_5 is comparatively a small number; for as we may readily foresee the two or three leading figures of it become zero. Now if in consequence of our having rejected an unwarrantable number of figures from the right of the numbers in X_5 , we had deprived ourselves of the means of getting any significant result for X_5 , in consequence of the remainder, as far as our numbers could influence it being zero, then, as remarked above, we should have concluded that there are at least two roots, each equal to the value of x in $X_4 = 0$, to as many figures as this value of x can be depended upon as accurate consistently with the result $X_5 = 0$.

To determine whether or not these roots are equal throughout, would require one or other of these three methods of proceeding:

1. We might correct the latter steps of the work, commencing at where the abbreviations are first introduced, and thus remove all doubt.

2. We might take the leading figure or figures furnished by the simple equation $X_1 = 0$, and extend these by Mr. HORNER's method of approximation, till we should arrive at that figure at which the roots separate, or till sufficient evidence is afforded that this point of separation is indefinitely postponed, and that therefore the roots are equal.*

3. Or finally, and which is generally the shortest course to our object, we might see if the two numbers in X_1 have the property mentioned in the foot-note at page 185. If they have not, the roots are unequal. If they have, then divide X by the square of X_1 , the common factors being expunged, and the roots under examination are equal or

* CAUCHY has determined a fractional expression, involving only the first coefficient of the proposed equation, and the one which is greatest in absolute value, which he proves to be less than the smallest difference between any two real and unequal roots. If, therefore, two or more roots, as the approximation proceeds, keep together up to a figure as small in the numerical scale as this fraction, we may infer that they must necessarily be equal, without extending the approximation farther. But unless the original coefficients are small, and the degree of the equation not beyond the fifth, the extreme smallness of this limit will render it we fear of but little value.

The expression referred to is

$$\frac{1}{A\{2(k+1)\}^{m-1}}$$

where A is the coefficient of the first term, k the greatest coefficient in absolute value, the coefficients being supposed whole numbers, and $m = \frac{1}{2}n(n-1)$, where n is the degree of the equation. (See *Journal de Mathématiques*, May 1838.) In connexion with this subject the student may also consult Note II. at the end of the present work, wherein it is shown that no difficulty can ever arise from the entrance of equal or nearly equal roots into the equation.

not according as the remainder of this division is zero or not.

We have here been supposing an extreme case. It is scarcely necessary to state that in practice the better plan will always be to secure at least four or five places of figures even for our last step; or so many that should the final or constant function become zero, we may immediately conclude the values of the roots indicated to three or four places of figures. It will be a very rare occurrence indeed for *unequal* roots to keep together throughout this extent, and then to differ at a more remote figure; yet, as the thing is not absolutely impossible, it is requisite that we know how to proceed should it be required to find, in addition to the nature and number of the roots and the leading figures of each, whether two or more remarkably close roots are really separable or not. But we shall not enter at greater length into these critical cases at present; we propose to resume the consideration of them in Note II. at the end, at which place it will be shown that there can never be the slightest perplexity attendant upon our method on account of the presence of equal roots, or of roots differing ever so little from each other.

(114.) We shall now give an example of the application of our method to an equation of the 6th degree, selecting for the purpose one which is adduced in a recent publication as evidence of the impracticability of STURM's Theorem in the higher equations.* The equation is this,

$$15. \quad x^6 + x^5 - x^4 - x^3 + x^2 - x + 1 = 0$$

whence

$$X_1 = 6x^5 + 5x^4 - 4x^3 - 3x^2 + 2x - 1,$$

* *Du Théorème de M. Sturm, et de ses Applications Numériques.* Par M. E. MIVY, Professeur au Collège de Nantes. Paris, 1836.

and the operation is as follows :

$$\begin{array}{r}
 \begin{array}{r}
 6-5 \quad +4 \quad +3 \quad -2 \quad +1 \\
 \times \\
 1+1 \quad -1 \quad -1 \quad +1 \quad -1+1
 \end{array} \\
 \hline
 \begin{array}{l}
 1 \quad -5 \quad +4 \quad +3 \quad -2 \quad +1 \\
 6 \quad 24+18-12+6 \\
 36 \quad -36-36+36-36+36
 \end{array} \\
 \hline
 17+14-27+32-37 \\
 \hline
 \begin{array}{r}
 17-14 \quad +27 \quad -32 \quad +37 \\
 \times \\
 6+5 \quad -4 \quad -3 \quad +2 \quad -1
 \end{array} \\
 \hline
 \begin{array}{l}
 1 \quad -14 \quad +27 \quad -32 \quad +37 \\
 102 \quad 2754-3264+3774 \\
 17^2 \quad -1156 \quad -867 \quad +578-289
 \end{array} \\
 \hline
 -1584+4104-4320+252 \\
 \text{or, } \div 4 \times 9 \\
 -44 \quad +114 \quad -120 \quad +7 \\
 \hline
 -44 \quad -114 \quad +120 \quad -7 \\
 \times \\
 17+14 \quad -27 \quad +32-37 \\
 \hline
 \begin{array}{l}
 -1277 \quad 145578-153240 \quad +8939 \\
 -22 \times 17 \quad -44880 \quad +2618 \\
 44 \times 22 \quad -26136 \quad +30976-35816
 \end{array} \\
 \hline
 -74562+119646+26877 \\
 \hline
 -74562-119646-26877 \\
 \times \\
 -44+114 \quad -120+7 \\
 \hline
 \begin{array}{l}
 -21 \cdot 69 \quad 2595 \quad +583 \\
 22 \quad -591 \\
 37281 \quad -4474 \quad +261
 \end{array} \\
 \hline
 2470 \quad -844 \\
 \text{or, } \div 2 \\
 1235 \quad -422
 \end{array}$$

Consequently the functions are as follow :

$$X = x^6 + x^6 - x^4 - x^3 + x^2 - x + 1$$

$$X_1 = 6x^5 + 5x^4 - 4x^3 - 3x^2 + 2x - 1$$

$$X_2 = 17x^4 + 14x^3 - 27x^2 + 32x - 37$$

$$X_3 = -44x^3 + 114x^2 - 120x + 7$$

$$X_4 = -74562x^2 + 119646x + 26877$$

$$X_5 = 1235x - 422$$

$$\therefore X_6 = -$$

$$X \quad X_1 \quad X_2 \quad X_3 \quad X_4 \quad X_5 \quad X_6$$

For $x = -\infty \dots + - + - - -$ three variations,

$= +\infty \dots + + + - - + -$ three " „

Hence the roots are all imaginary. And this we might have inferred at once without writing down the functions, because there are always as many pairs of imaginary roots as there are variations in the leading signs. (*Theory of Equations*, p. 144.)

(115.) The following equation of the seventh degree is from FOURIER (page 111).

16. Required the number and situations of the real roots of the equation

$$x^7 - 2x^6 - 3x^5 + 4x^3 - 5x + 6 = 0.$$

Here the leading functions are

$$x^7 + 0x^6 - 2x^5 + 0x^4 - 3x^3 + 4x^2 - 5x + 6$$

and

$$7x^6 + 0x^5 - 10x^4 + 0x^3 - 9x^2 + 8x - 5,$$

* In the work of M. MINY, before referred to, the entire series of functions after this are erroneous; but the character of the roots is nevertheless correctly determined.

and the following is the numerical operation,

$$\begin{array}{r}
 7+0+10+0+9-8+5 \\
 \times \\
 1+0-2+0-3+4-5+6 \\
 \hline
 0 \quad 10+0+9-8+5 \\
 1 \quad -14+0-21+28-35+42 \\
 7 \quad \hline
 4+0+12-20+30-42 \\
 \text{or, } \div 2 \\
 2+0+6-10+15-21 \\
 \hline
 2-0-6+10-15+21 \\
 \times \\
 7-0-10-0-9+8-5 \\
 \hline
 0 \quad -42+70-105+147 \\
 7 \quad -20-0-18+16-10 \\
 2 \quad \hline
 62-70+123-163+10 \\
 \hline
 62+70-123+163-10 \\
 \times \\
 2+0+6-10+15-21 \\
 \hline
 35 \quad 2450-4305+5705-350 \\
 31 \quad -3813+5053-310 \\
 31^s \quad 5766-9610+14415-20181 \\
 \hline
 -4403+8862-19810+20531 \\
 \hline
 -4403-8862+19810-20531 \\
 \times \\
 62-70+123-163+10 \\
 -54 \cdot 81672 \quad 485786-1085919+1125442 \\
 -62 \quad -1228220+1272922 \\
 4403 \quad 541569-576793+44030 \\
 \hline
 200865+489790-1169472 \\
 \hline
 200865-489790+1169472 \\
 \times \\
 -4403+8862-19810+20531 \\
 \hline
 19615 \quad -96072+229392 \\
 -4403 \quad -51492 \\
 200865 \quad -39791+41240 \\
 \hline
 187355-270632
 \end{array}$$

Hence the functions are

$$X = x^7 - 2x^5 - 3x^3 + 4x^2 - 5x + 6$$

$$X_1 = 7x^6 - 10x^4 - 9x^2 + 8x - 5$$

$$X_2 = 2x^5 + 6x^3 - 10x^2 + 15x - 21$$

$$X_3 = 62x^4 - 70x^3 + 123x^2 - 163x + 10$$

$$X_4 = -4403x^3 + 8862x^2 - 19810x + 20531$$

$$X_5 = 200865x^2 + 489790x - 1169472$$

$$X_6 = 187355x - 270632$$

$$\therefore X_7 = +$$

As there are two variations in the signs of the leading terms, we infer that the equation has two pairs of imaginary roots, but not a greater number: there are therefore *three* real roots; and since the variations in the final signs exceed those in the leading signs by *two*, it follows that two of the roots are positive.

Since the greatest negative coefficient in the proposed equation is 5, it follows (*Theory of Equations*, p. 38,) that 6 is a superior limit to the positive roots; so that our substitutions may be confined to numbers below 6.

For $x = 0 \dots + - - + + - - +$ *four variations*

$= 1 \dots + - - - + - - +$ *four* „

$= 2 \dots + + + + - + + +$ *two* „

Consequently the two positive roots are both situated between 1 and 2.

FOURIER has not succeeded in discovering the character of the roots of the foregoing equation. He seems to imply that *five* of them are real instead of *three* only.*

As to the negative root of the equation, we easily find, by changing the alternate signs, and writing it thus,

$$x^7 - 2x^6 - 3x^5 - 4x^4 - 5x^3 - 6 = 0,$$

that $x = 1$, and $x = 2$, give opposite signs; the negative root is therefore between -1 and -2 .

(116.) The equation just analyzed, though of the seventh degree, is not complete in all its terms. It is easy to see that this circumstance operates favorably in restraining the magnitude of the numerical results, especially those of them which are more immediately dependent on the second term of the proposed equation. We shall now take a complete equation of the seventh degree; and one, moreover, in which the leading term has a coefficient greater than unity.

17. Let the equation

$$4x^7 - 6x^6 - 7x^5 + 8x^4 + 7x^3 - 23x^2 - 22x - 5 = 0$$

be proposed.

Then the functions X , X_1 are

$$X = 4x^7 - 6x^6 - 7x^5 + 8x^4 + 7x^3 - 23x^2 - 22x - 5$$

$$X_1 = 28x^6 - 36x^5 - 35x^4 + 32x^3 + 21x^2 - 46x - 22;$$

and the following is the numerical operation, the abbreviations being first introduced at the commencement of the third step of the work.

* *Analyse des Equations*, pages 111 and 153.

$$\begin{array}{r}
28+36+35-32-21+46+22 \\
\begin{array}{r} \times \\ 4-6-7+8+7-23-22-5 \end{array} \\
-3-108-105+96+63-138-66 \\
14 \quad 490-448-294+644+308 \\
98-686+784+686-2254-2156-490 \\
\hline
304-231-488+1547+1986+556 \\
304+231+488-1547-1986-556 \\
\begin{array}{r} \times \\ 28-36-35+32+21-46-22 \end{array} \\
-1119-253489-546072+1731093+2222334+622164 \\
2128 \quad 1038464-3292016-4226208-1183168 \\
76 \times 304 \quad -808640+739328+435184-1062784-508288 \\
\hline
28665+3098760+2009931+23618-113876 \\
28665-3098760-2009931-23618+113876 \\
\begin{array}{r} \times \\ 304-231-488+1547+1986+556 \end{array} \\
-33059 \cdot 2937 \quad 102442817+66446899+780794-3764660 \\
304 \quad -611019 \quad -7180+34618 \\
28665 \quad -13989+44345+56929+15938 \\
\hline
-101817809-66409734-872341+3748722 \\
-101817809+66409734+872341-3748722 \\
\begin{array}{r} \times \\ 28665+3098760+2009931+23618-113876 \end{array} \\
-3080074-20454689-268687+1154634 \\
-28665 \quad -2501+10746 \\
101817809 \quad 20464677+240473-1159460 \\
\hline
-7487+17468+4826 \\
-7487-17468-4826 \\
\begin{array}{r} \times \\ -1018178-664097-8723+37487 \end{array} \\
3040 \quad -5310-1467 \\
1018 \quad -491 \\
7 \quad -6+26 \\
\hline
5807+1441 \\
\text{or } \div 11 \\
528+131.
\end{array}$$

This last function, containing x only in the first degree, gives, when equated to zero,

$$x = -\frac{131}{526}.$$

If this be substituted for x in the preceding function of the second degree, namely, in

$$-7487x^2 + 17468x + 4826,$$

the result of the first two terms will be -4795 ; hence the whole result will be $+31$; consequently, the final function takes the sign *minus*.

We shall have no occasion to write down the several functions to which the foregoing sets of coefficients belong; for it is at once seen, from the signs of the leading terms in those functions, that the equation has only one real root, and the place of this is readily determined from the original function X .

The last term in this function being *minus*, the root sought is of course *plus*. It is found, in the usual way, to lie between 2 and 3.

(117.) It will have been observed that, in all the preceding examples, the derived functions descend regularly, in degree, by unity. No instance has occurred in which two consecutive functions differ in degree by two or more units. Such instances, indeed, are by no means common; for an equation of the n th degree usually furnishes $n+1$ functions, X, X_1, X_2 , &c. Nevertheless it sometimes happens that the functions are fewer than $n+1$ in number, owing to an interruption taking place in the regularity with which the degrees of these functions ordinarily dimi-

nish. In every such case a corresponding interruption to the regularity of our uniform process of deriving these functions will occur; so that, commencing at the point where the break happens, the step next in advance must be executed in a manner somewhat out of harmony with the prevailing mode of procedure.

This circumstance, however, will not at all add to the complication or difficulty of the process; on the contrary, the oftener these interruptions take place, the shorter and easier will our work become; but it is very important to keep in remembrance the necessary modification which our method is to undergo at these stages.

What this modification is, is soon discovered from inspecting the general formulas at the commencement of this dissertation. Those formulas were constructed on the express condition that the several remainders (see page 142) descend regularly in degree by unity, when the proper powers of x are inserted; and we have now to ascertain, when two consecutive remainders differ by *two* units, how we are to derive from them the third remainder.

If

$$ax^n + bx^{n-1} + cx^{n-2} + \dots \quad [A]$$

be the first remainder, and

$$a'x^{n-2} + b'x^{n-3} + c'x^{n-4} + \dots \quad [B]$$

the second, the third must be of the form

$$b''x^{n-3} + c''x^{n-4} + \dots$$

to obtain which will evidently require *three* terms in the

quotient of [A] by [B] instead of two, as in the case considered at page 143, the third term being chosen so as to cause the third term in the equation [C] at page 143 to vanish, in addition to the two preceding terms. This third term, as far at least as the coefficient is concerned, is obviously $\frac{a''}{a'}$; or, to avoid fractions, if we first multiply [C] by a' , the third term of the quotient will be a'' ; and, therefore, the remainder sought arises thus:

$$\begin{array}{r}
 a'a''x^{n-2} + a'b''x^{n-3} + a'c''x^{n-4} + \dots \\
 a''a'x^{n-2} + a''b'x^{n-3} + a''c'x^{n-4} + \dots \\
 \hline
 0 \quad + \quad b'''x^{n-3} + c'''x^{n-4} + \dots
 \end{array}$$

Consequently, in the case supposed, this is the supplementary operation which must be introduced to complete the step. Hence we must proceed in the usual way, that is, as at page 144, to the remainder

$$a'' + b'' + c'' + \dots$$

we may then bring down, under this, the preceding remainder

$$a' + b' + c' + \dots$$

and, multiplying $b'' + c'' + \dots$ by a' , and $b' + c' + \dots$ by a'' , we may arrange the products under one another, and subtract the lower set from the upper.

This appears to us to be the most convenient way of arranging the additional work, and that which best unites with the general process. Instead of actually *subtracting*, however, we shall, as in the other parts of the work,

convert the operation into one of addition by changing the signs.

(118.) An example will sufficiently illustrate the numerical operation.

Let the two consecutive functions or remainders be

$$4x^5 - 3x^4 + 0x^3 + 2x^2 - x + 1$$

and

$$2x^3 + 0x^2 - x - 4.$$

Then executing the work of an ordinary step first, and afterwards supplying the necessary addition, the whole will stand thus :

$$\begin{array}{r}
 2+0 \quad +1+4 \\
 \times \\
 4-3 \quad +0+2 \quad -1+1 \\
 \hline
 3 \left| \begin{array}{l} 0 \quad -3-12 \\ 4 \quad 4+16 \\ 2 \quad 0 \quad +4 \quad -2 \quad +2 \end{array} \right. \\
 -4 \left| \begin{array}{l} -17+14 \quad -2 \\ 2 \quad +0 \quad +1 \quad +4 \end{array} \right. \\
 \hline
 -34+28 \quad -4 \\
 0 \quad -4-16 \\
 \hline
 -34+24-20
 \end{array}$$

The result $-4 - 17 + 14 - 2$, which closes the ordinary step, has all its signs changed; also the terms $+0 + 1 + 4$ have had the signs changed: but the multiplication of these by the -4 , the sign of this having been changed,

restores the signs of the product $0 - 4 - 16$ to what it would have been had none of these changes been made. It appears, therefore, that of the two rows of terms

$$\begin{array}{r} - 34 + 28 - 4 \\ 0 - 4 - 16 \end{array}$$

the first has all its signs the opposite of those really belonging to it, and the second has its proper signs, which become, in effect, changed by adding the lower row instead of subtracting it, and thus our final result

$$- 34 + 24 - 20$$

is the true remainder with changed signs; and it is with such change that, in STURM's theorem, we wish the remainders to present themselves. The final result therefore under the prescribed arrangement harmonizes with all the other results of the process.*

As the multipliers $- 4$ and 2 have the common factor 2 , this factor may evidently be suppressed; and thus, like as with the other multipliers, the operation simplified. The final result, when thus reduced, will therefore be

$$- 17 + 12 - 10$$

To render the method still plainer we shall give another example.

* It is easy to see that the entire operation of STURM's theorem might be conducted according to this plan, every successive function being obtained at two steps instead of at one. The intermediate steps, in this mode of proceeding, would furnish the intermediate or auxiliary remainders; but the method employed throughout this dissertation will in general be found to require both less labour and less space.

Suppose the two consecutive functions or remainders to be

$$2x^4 - 6x^3 + 3x^2 + x - 2$$

and

$$-3x^3 + 2x - 4$$

$$\begin{array}{r}
 \begin{array}{r}
 -3-2 \quad +4 \\
 \times \\
 2-6 \quad +3 \quad +1-2 \\
 \hline
 \end{array} \\
 \begin{array}{r}
 14 \overline{) \begin{array}{r} -28 \quad +56 \\ -6 \quad -24 \\ 9 \quad 27 \quad +9 \quad -18 \end{array}} \\
 \hline
 \begin{array}{r}
 25 \overline{) \begin{array}{r} -65 \quad +18 \\ -3 \quad -2 \quad +4 \end{array}} \\
 \hline
 \begin{array}{r}
 130 \quad -54 \\
 -50 \quad +100 \\
 \hline
 80 \quad +46 \\
 \text{or } \div 2 \\
 \hline
 40 \quad +23 \\
 \hline
 \end{array}
 \end{array}$$

(119.) If the proposed functions differ in degree by *three* units instead of two, then four terms in the result [C], p. 143, must be rendered zero; two are made zero by the usual step; so that we shall only have to employ the result of this step, in conjunction with the original divisor, and to proceed with them in the ordinary way, in order to render two more terms zero; that is, in order to arrive at the desired remainder. We shall, however, as before, employ

addition instead of subtraction, that this remainder may appear with changed signs. The following example will sufficiently illustrate the method of proceeding.

Let the two functions be

$$2x^6 - 3x^5 + 2x^4 + x^3 - 3x^2 - 3x + 4$$

and

$$4x^3 - 6x^2 + x + 1$$

$$\begin{array}{r}
 4+6 \quad -1 \quad -1 \\
 \times \\
 2-3 \quad +2 \quad +1 \quad -3-3+4 \\
 \hline
 0 \quad \left[\begin{array}{l} -1 \quad -1 \\ 4 \quad +2 \quad -6 \quad -6+8 \\ -3 \quad -1 \quad +6 \quad +6-8 \end{array} \right. \\
 \hline
 4+6 \quad -1 \quad -1 \\
 \times \\
 -3-1 \quad +6 \quad +6 \quad -8 \\
 \hline
 -11 \quad \left[\begin{array}{l} -66+11+11 \\ 6 \quad +6 \\ 48+48-64 \\ -12+65-53 \end{array} \right. \\
 \hline
 \hline
 \end{array}$$

Hence the remainder, with changed signs, is

$$-12x^2 + 65x - 53$$

It is scarcely necessary to remark, that if our final re-

mainder is to be still further postponed, till it reach one degree lower, as will happen when the functions differ in degree, by four units, then we are to continue the work as in the former examples, and so on. Whenever the sought remainder is deduced, a thick line, or rule, should be drawn under it, to mark distinctly the close of a step.*

(120.) We shall now give an example in which STURM's functions do not regularly descend in degree by unity, and in the working of which the modification just explained will therefore be necessary. This example, like the last, is from FOURIER.

18. Required the number and situations of the real roots of the equation

$$x^6 - 12x^5 + 60x^4 + 123x^3 + 4567x - 89012 = 0$$

* The above example, as well as the one first given, is from the *Penny Cyclopædia*, article INVOLUTION. The writer of that article, we believe Mr. DE MORGAN, gives a method of deducing the remainder, which is somewhat shorter than that which we have recommended, and recommended chiefly because of its more readily uniting with our general numerical process. Mr. DE MORGAN'S method of conducting and arranging the work may perhaps be preferred by some, and we therefore refer to his paper, as also to remark that the operations in his method, in that of M. MINY, and in the steps employed in the present Dissertation, are all suggested by one and the same general principle, and that a principle of extreme obviousness and simplicity. It is solely in the mode of applying this principle to actual practice, that is, in the mere numerical arrangement of the work, that the labour involved in it is to be economized, and reduced to its least possible amount.

$$\begin{array}{r}
6+60-240 \quad +0 \quad -246 \quad -4567 \\
\begin{array}{r} \times \\ 1-12 \end{array} +60 \quad +0 \quad +123 \quad +4567-89012 \\
\hline
-2 \left| \begin{array}{r} -120+480 \quad +0 \quad +492 \quad +9134 \\ 1-240 \quad +0-246-4567 \\ 6 \quad 360 \quad +0+738+27402-534072 \end{array} \right. \\
\hline
0-480-492-23327+524938 \\
\hline
-480+492 \quad +23327-524938 \\
\begin{array}{r} \times \\ 6-60 \end{array} -240 \quad -0 \quad -246-4567 \\
\hline
441 \left| \begin{array}{r} 216972+10287207 \quad - \quad 231497658 \\ -40-933080+20997520 \end{array} \right. \\
3200 \left| \begin{array}{r} 768000 \quad +0 \quad +787200 \quad +14614400 \\ * -51892 \left| \begin{array}{r} -31284727 \quad +230710458 \quad -14614400 \\ -480 \left| \begin{array}{r} +492 \quad +23327 \quad -524938 \end{array} \right. \end{array} \right. \\
\hline
150167 \quad -1107410 \quad +70149 \\
-255 \quad -12105 \quad +272401 \\
\hline
149912 \quad -1119515 \quad +342550 \\
\hline
149912+1119515-342550 \\
\begin{array}{r} \times \\ -480-492-23327 \end{array} +524938 \\
\hline
-4076 \cdot 8 \left| \begin{array}{r} -4564 \quad +1397 \\ -480 \quad -164 \\ 1499 \quad -3 \quad +80 \end{array} \right. \\
\hline
4731-1477
\end{array}$$

* The two numbers in this side column, and which are employed as multipliers for the completion of the step before us, agreeably to the directions already given at (118), have a common factor, 4. Expanding this, therefore, we might have employed as multipliers, the numbers 12973, and 120. We have preserved the original numbers in order to render the *method* of proceeding the more plain; but in practice it will always be advisable to substitute for the numbers themselves those which arise from expanding the common factor.

Consequently, STURM's functions are as follow :

$$X = x^6 - 12x^5 + 60x^4 + 123x^3 + 4567x - 89012$$

$$X_1 = 6x^5 - 60x^4 + 240x^3 + 246x + 4567$$

$$X_2 = -480x^4 - 492x^3 - 23327x + 524938$$

$$X_3 = 149912x^3 - 1119515x + 342550$$

$$X_4 = 4731x - 1477$$

$$\therefore X_5 = +$$

We conclude at once, and without even writing down the functions, that the equation has *four* imaginary roots; one pair being indicated by the absence of the function of the fourth degree, and another pair by the leading sign of X_5 . There are thus only two real roots; and since, moreover, the variations in the final signs exceed those in the leading signs by *one*, it follows that one of these two roots is positive, and the other negative. Their situations therefore may be readily determined from the single function X , the others being useless for this purpose.

To find a superior limit to the positive root, we have the inequality

$$1 + \frac{60}{x^4} + \frac{123}{x^3} + \frac{4567}{x^2} > \frac{12}{x} + \frac{89012}{x^6},$$

which is satisfied for $x = 7$, and for every higher number, but not for $x = 6$; hence the positive root is between 6 and 7.

For the negative root, change the alternate signs of X , taking account of the term $0x^3$, writing it thus,

$$x^4 + 12x^3 + 60x^2 + 123x^2 - 4567x - 89012,$$

and it will soon be found that the consecutive numbers, which give opposite signs to this function, are $x = 5$ and $x = 6$. Consequently, the negative root lies between -5 and -6 .

(121.) Enough has now been done to put the student fully in possession of what appears to us to be the best method of conducting the numerical operations in the practical application of STURM'S THEOREM; and, at the same time, to enable him to institute a fair comparison between this method and the methods of BUDAN and FOURIER. To render such a comparison the easier, we have selected all our more difficult examples from the work of FOURIER. Some of these, however, are not fully analyzed by FOURIER. The examination, for instance, of the equation of the seventh degree, which we have considered at p. 191, is left by FOURIER incomplete; and his results are, in consequence, presented in an imperfect, if not indeed in an erroneous form. We are not aware whether or not STURM's theorem has ever till now been applied to equations of the sixth and seventh degrees; but the fact here established of its ready application to such equations, and equations too whose coefficients are by no means remarkably small, will serve to moderate the objections which have been made to the theorem on the ground of its general impracticability in equations of a degree higher than the fourth.

By conducting the operation as at first proposed by STURM, that is, by going through the ordinary routine of seeking the common measure of two algebraic functions,

the work would, no doubt, become both fatiguing and cumbersome in equations of the fifth and sixth degrees.

But by employing only detached coefficients, as in HORNER's method of solution, suppressing the useless quotients, precluding fractions, and rejecting the redundant figures in the closing steps, or rather providing against their entrance—by combining all these facilities, and disposing of the whole work so as to effect the greatest saving in numerical labour, we have seen that an equation of even the *seventh* degree may be completely analyzed in reasonable space, and with but a moderate amount of arithmetical calculation.

(122.) Although the theorem of STURM is thus reduced to a practicable form, and is, on account of its infallibility, invaluable in the analysis of numerical equations, yet the rules of FOURIER, or rather the transformations of BUDAN, are not without their use. In equations of a high order it will frequently be found that these transformations, which may always be effected with comparatively little trouble, will furnish us with useful preliminary information as to the partial construction of the equation; and in favorable cases the entire constitution of it will thus be made known. Any foresight into the nature of the roots, before actually applying STURM's theorem, is of advantage, either in the way of checking our numerical results, or of enabling us to anticipate or to dispense with, the closing steps of the work. As, for instance, if a preliminary examination of an equation by the rules and tests given by BUDAN, shows us that there is but one doubtful interval, and we afterwards, in the work of deriving STURM's functions, obtain a result with its leading sign *minus*, we may conclude at once that the equation has two, and only two, imaginary roots.

If we had submitted the equation at p. 203 to these tests, we should have found two doubtful intervals; or, at

least, one interval doubtful, and one indicating a pair of imaginary roots; and, as the result of the very first step in STURM's process shows us, that there are at least *two* pairs of imaginary roots, we might have stopped the work at that step, and have affirmed with confidence, that the two unambiguous roots, made known by BUDAN's transformations, are all that exist in the equation.

Many instructive examples of the value and efficiency of BUDAN's rules, in certain cases, are given in Mr. LOCKHART's *Resolution of Equations*, before referred to; and besides this latter work, the student may consult with advantage, on the subject of these rules and criteria, the second edition of BUDAN's own treatise,* a paper by M. VINCENT, on the Solution of Numerical Equations in the *Journal de Mathématiques*, for October 1836, Professor DAVIES's *Key to Hutton's Mathematics*, and the fourth chapter of the *Theory of Equations*, by the author of the present work.

(123.) We have now to notice a short and easy method of performing the first step in STURM's process; that is, of obtaining the function X_1 .

Abridgment in this early stage of the work is, we must acknowledge, of comparatively but little importance in equations of elevated degrees; still in those of the third and fourth degrees, which, after all, occur more frequently than those of the higher orders, the saving of trouble effected by this simplification of the general method, although extending only to the first step, does really bear a perceptible proportion to the entire work. In such equations therefore, at least, we think the improvement to be noticed should always be introduced; and even in equations of more advanced degrees, whose coefficients are

* Nouvelle Méthode pour la Résolution des Equations Numériques. Paris, 1822.

large numbers, this more brief and rapid method of performing the first step is well worth employing.

The improvement we have in view is derived from the general process, in the following manner:

(124.) Let

$$X = ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + \dots$$

then

$$X_1 = nax^{n-1} + (n-1)bx^{n-2} + (n-2)cx^{n-3} + \dots$$

and, proceeding according to the rule already explained, X_2 is determined as follows:

$na - (n-1)b$	$-$	$(n-2)c$	$-$	$(n-3)d$	$- \dots$
$\times$					
$a + b$	$+$	c	$+$	d	$+ \dots$
b	$ $	$-(n-1)b^2$	$-$	$(n-2)bc$	$-$
na	$ $	$-n(n-2)ac$	$-$	$n(n-3)ad$	$-$
n^2a	$ $	n^2ac	$+$	n^2ad	$+$
n^2a	$ $	n^2ae	$+$	n^2ae	$+ \dots$
$\{(n-1)b^2 - 2nac\} + \{(n-2)bc - 3nad\} + \{(n-3)bd - 4nae\} + \dots$					

Consequently

$$X_2 = \left(\begin{array}{c} (n-1)b^2 \\ -2nac \end{array} \right) x^{n-2} + \left(\begin{array}{c} (n-2)bc \\ -3nad \end{array} \right) x^{n-3} + \left(\begin{array}{c} (n-3)bd \\ -4nae \end{array} \right) x^{n-4} + \dots$$

And from this general form is immediately derived the following mode of computing the coefficients of X , in any particular case, namely,

Arrange the coefficients of X , in a row, with their proper signs. Underneath these write the coefficients of X , commencing with the *second*, after having multiplied them in order by the numbers 1, 2, 3, 4, &c., taking care however

to put down all the results of these multiplications, except the first, with changed signs. We shall thus have two rows of figures of equal extent.

Cut off the leading term in each row for multipliers, and combine them with the other figures, as at page 199; that is, the under multiplier with the upper row, and the upper multiplier with the under row; and the step will be completed by adding the results.

NOTE. If the second term of X be zero, that is, if $b = 0$, in the preceding general expression for X_2 , the coefficients in that expression will be reduced simply to

$$X_2 = -2nac - 3nad - 4nac - \dots$$

or, still more simply, to

$$-2c - 3d - 4e - \dots$$

so that to obtain the coefficients in this case we shall merely have to multiply the coefficients of X , commencing with the third, (or the *second* that actually appears) by -2 , -3 , -4 , &c. respectively.

It is scarcely necessary to add that, like as in the general process, if the multipliers have a common factor, it is to be expunged.

And moreover, if the numbers in the lower row, that is, the coefficients, after the first, in the proposed equation, are found to have a common factor when the multiplications by 1, 2, 3 have been performed, that factor may be neglected.

(125.) We shall apply these rules to the examples given in the *Theory of Equations*; as the work for the common measure is not there exhibited at length, except in one instance.

1. Let the proposed equation be

$$x^3 - 5x^2 + 8x - 1 = 0$$

$$\therefore X_1 = 3x^2 - 10x + 8,$$

and by the rule

$$\begin{array}{r} 3 \overline{) -10 \ +8} \\ -5 \overline{) -16 \ +3} \\ \hline 50 -40 \\ -48 \ +9 \\ \hline 2 -31 \\ \hline \end{array}$$

$$\therefore X_2 = 2x - 31$$

2. Let the equation be

$$x^3 + 11x^2 - 102x + 181 = 0$$

$$\therefore X_1 = 3x^2 + 22x - 102,$$

and proceeding by the rule

$$\begin{array}{r} 3 \overline{) +22 -102} \\ 11 \overline{) +204 -543} \\ \hline 242 -1122 \\ 612 -1629 \\ \hline 854 -2751 \\ \hline \end{array}$$

$$\therefore X_2 = 854x - 2751.$$

(126.) From these two examples the student cannot fail to recognize the extreme simplicity to which everything connected with the complete solution of a *cubic equation*

is now reduced. The means by which X_2 , and thence the sign of the final function X_3 , is obtained, are so easy, and the computations so very trifling, that one would scarcely be prepared to expect that such simple operations actually comprehend all the work essentially necessary in seeking the common measure of X and X_1 .

In this improved method indeed all analogy to the original course of proceeding seems to have disappeared; and it might be taken for an entirely independent rule for separating the roots of cubic equations. In conjunction with Mr. HORNER's process of continuous approximation to the roots, nothing farther can surely be desired in the way of simplifying the solution of equations of the third degree; although but a few years ago the complete analysis of this class of equations was numbered among the difficulties of algebra.

3. Let the equation be

$$x^3 - 7x + 7 = 0,$$

which is an equation of some celebrity on account of the difficulty attending the separation of the roots by the older methods.*

Here,

$$X_1 = 3x^2 - 7,$$

and by the NOTE

$$X_2 = 14x - 21,$$

or

$$2x - 3$$

$$\therefore X_3 = +,$$

and the roots may now be separated with the greatest ease.

* See LAGRANGE. *Traité de la Résolution des Equations Numériques*, pp. 33 and 237. 1826.

4. As another example of this kind, let the equation be

$$8x^3 - 6x - 1 = 0.$$

As before, the functions may be written at once from mere inspection, without any numerical work ; they are

$$X = 8x^3 - 6x - 1$$

$$X_1 = 4x^2 - 1$$

$$X_2 = 12x + 3$$

or

$$4x + 1$$

$$\therefore X_3 = +.$$

As a last application of our method to cubic equations, we shall take the following, which has been selected by Mr. LOCKHART on account of its difficulty ; and which that gentleman has analysed at length, by a combination of the methods of NEWTON and BUDAN, in the work before referred to.*

$$5. \quad X = 12x^3 - 120x^2 + 326x - 127 = 0$$

$$X_1 = 36x^2 - 240x + 326.$$

Consequently, by the rule,

* *Resolution of Equations*, p. 26. Oxford, 1837.

$$\begin{array}{r|rr}
 & 3^9 & & \\
 36 & & -240 & + 326 \\
 -120 & & -652 & + 381 \\
 \hline
 & & 2400 & - 3260 \\
 & & -1956 & + 1143 \\
 \hline
 & & 444 & - 2117 \\
 \hline
 \end{array}$$

$$\therefore X_1 = 444x - 2117$$

$$\therefore X_2 = -.$$

Hence the proposed equation has *two* imaginary roots. And in this manner may the character of the roots of a cubic equation always be ascertained with very little trouble. We shall proceed to biquadratic equations.

6. Let the equation be

$$X = x^4 - 2x^3 - 7x^2 + 10x + 10 = 0$$

$$\therefore X_1 = 2x^3 - 3x^2 - 7x + 5,$$

and by the rule

$$\begin{array}{r|rrrr}
 2 & -3 & -7 & +5 & \\
 -2 & +14 & -30 & -40 & \\
 \hline
 & 6 & +14 & -10 & \\
 & 28 & -60 & -80 & \\
 \hline
 & 34 & -46 & -90 & \\
 \hline
 \end{array}$$

* As 36 and 120 are both divisible by 12, leaving for quotients 3 and 10, we have employed these for multipliers instead of the original numbers; and have placed them in small characters over the latter. But in practice it will be better to expunge these, putting the smaller numbers in their place.

As the multipliers here have a common factor 2, we ought to have expunged it, and to have employed the multipliers 1 and -1 . When properly reduced therefore the operation is as follows:

$$\begin{array}{r}
 1 \mid -3 \quad -7 \quad +5 \\
 -1 \mid +14 \quad -30 \quad -40 \\
 \hline
 3 \quad +7 \quad -5 \\
 14 \quad -30 \quad -40 \\
 \hline
 17 \quad -23 \quad -45 \\
 \hline
 \end{array}$$

$$\therefore X_2 = 17x^2 - 23x - 45.$$

We might now, commencing with this function and the preceding one X_1 , make the determination of X_3 , the first step of the general rule; and this indeed is the only step requisite. To separate the roots of a biquadratic equation therefore can never be a work of much labour or difficulty. But this will be still more distinctly shown in NOTE II. at the end.

As an application of the precept in the Note to an equation of the fourth degree, let us take

$$7. \quad X = 2x^4 - 13x^2 + 10x - 19$$

$$\therefore X_1 = 4x^3 - 13x + 5$$

and by the NOTE

$$X_2 = 26x^2 - 30x + 76$$

or

$$13x^2 - 15x + 38$$

and since in this latter function

$$4 \times 13 \times 38 > 15^2$$

the operation terminates here.

As a final example we shall now show the amount of abridgment effected by our rule in the first step of the operation when the equation is one of the sixth degree; taking for that purpose the equation at page 203, namely,

$$8. x^6 - 12x^5 + 60x^4 + 0x^3 + 123x^2 + 4567x - 89012 = 0$$

$$\therefore X_1 = 6x^5 - 60x^4 + 240x^3 + 0x^2 + 246x + 4567,$$

and by the rule

6	-60+240	+ 0	+246	+4567
-12	-120	-0	-492	-22835+534072
	120-480	- 0	-492	-9134
	-120	-0	-492	-22835+534072
	0-480	-492	-23327	+524938

Consequently

$$X_2 = -480x^4 - 492x^3 - 23327x + 524938.$$

And commencing now with this function, and the preceding one X_1 , we may, as at page 204, proceed with the process by the general method.

(127.) We here terminate what we had to deliver upon the subject of STURM's Theorem; simply adding two NOTES, devoted to some details essentially necessary to the completion of our purpose in the present Essay, but which we have thought it desirable to postpone till the end.

The first of these notes briefly points out a relation between STURM's coefficients and those which appear in

LAGRANGE's equation of the squares of the differences ; and the second enters into an examination of the efficacy of the abbreviated method in those very delicate cases, where roots, either accurately equal or differing by only a minute quantity, exist in the equation.

The latter of these cases, namely, that in which several roots differ but very little from one another, is that which invariably proves a source of perplexity, if not of insurmountable difficulty, whenever the analysis of the equation is attempted by any method except that furnished by STURM. And it may at first seem questionable whether even that method, when modified and curtailed as we have recommended, may not allow some particulars to elude detection, which the unabridged, but more laborious process, would fully develope. It is of consequence therefore to prove that our abbreviations may be as safely employed in these critical cases as in those wherein the roots have no remarkable proximity to one another. Note II. is chiefly occupied with establishing this truth.

NOTE I. (page 170).

On the Equation of the Squares of the Differences, in connexion with Sturm's Functions.

(128.) THERE is a very intimate connexion between the coefficients of STURM's functions and those of the equation of the squares of the differences proposed by WARING and LAGRANGE; both, in fact, resting ultimately on the same principle, namely, the principle involved in the process for seeking the common measure of two algebraic functions.

In the methods of WARING and LAGRANGE, however, the number of conditions, for determining the character of the roots of an equation, speedily becomes so great and so complicated in their nature, as to be of little practical value in the solution of numerical equations. As remarked in the text, the conditions for determining the character of the roots of an equation of the fifth degree are ten in number. They are exhibited at length by WARING, in the *Philosophical Transactions* for 1763. This eminent analyst was unquestionably the first who proposed the equation of the squares of the differences as a means for discovering the number of real and imaginary roots in any proposed equation.

The ten conditions which WARING showed to be implied in this equation of the squares of the differences, when the proposed is of the fifth degree, are also given by LAGRANGE at page 108 of the work referred to; a glance at them will be sufficient to show that their determination must involve an enormous amount of numerical labour.

LAGRANGE, however, with his usual penetration, suspected that some of these conditions might be embodied among the others; and there is little doubt, if everything absolutely unessential about them could be ascertained and suppressed, that we should find the resulting conditions thus modified to be the same in effect as those implied in STURM's functions.

(129.) LAGRANGE's conditions for the equation of the *fourth* degree,

$$x^4 + Bx^3 - Cx + D = 0,$$

are as follow,

$$a = -8B$$

$$b = 22B^2 + 8D$$

$$c = -18B^3 + 16BD + 26C^2$$

$$d = 17B^4 + 24B^2D - 7.16D^2 + 3.16BC^2$$

$$e = -4B^5 - 2.27C^2B^3 - 8.27C^2D + 3.4^3BD^2 - 2.4^3B^2D$$

$$f = 4^4D^3 - 2^2.4^2B^3D^2 + 4^2.3^2C^2BD + 4^2B^4D - 4C^3B^3 - 3^3C^4.$$

If all the quantities a, b, c, d, e, f , are positive, the roots will all be real; and they will all be imaginary if the quantity f is positive, and any of the others negative. Now LAGRANGE finds that f being positive, all the other quantities will be so too, provided there exist at the same time

$$B < 0, \text{ and } B^2 - 4D > 0.$$

These, therefore, in conjunction with the condition $f > 0$,

are all the conditions essentially necessary to warrant the inference that the roots are all real.

In order that some one of the quantities a, b, c, d, e may be negative, and consequently the four roots imaginary, the only essential condition is

$$B > 0, \text{ or } B^2 - 4D < 0.$$

In the solution adopted in our general example of the fourth degree in the text, $a = B, b = -C, c = D$; making therefore these substitutions in the expression for f , we have

$$f = 4^4c^3 - 2^3.4^2a^2c^2 + 4^2.3^2b^2ac + 4^2a^4c - 4b^2a^2 - 3^2b^4,$$

and the first pair of conditions above is

$$a < 0, \text{ and } a^2 - 4c > 0 \dots [1]$$

which must exist, in conjunction with $f > 0$, if the four roots are real.

Now, by referring to STURM's functions, page 166, we see that in order that the roots may all be real, we must have the conditions.

$$X_4 = 4cA^2 - 3bAB + 2aB^2 > 0,$$

$$a < 0, \text{ and } 8ac - 2a^2 - 9b^2 > 0 \dots [2]$$

Computing the first of these expressions, in terms of the coefficients a, b, c we find it to be

$$4^4a^2c^3 - 2^3.4^2a^4c^2 + 4^2.3^2b^2a^2c + 4^2a^6c - 4b^2a^2 - 3^2a^2b^4,$$

or, dividing by the essentially positive quantity a^2 ,

$$4^4c^3 - 2^2.4^2a^2c^2 + 4^2.3^2b^2ac + 4^2a^4c - 4b^2a^3 - 3^2b^4,$$

which is the very same as LAGRANGE's f .

(130.) But the third condition [1], determined by LAGRANGE, is undoubtedly simpler than that [2] immediately given by STURM; yet the latter is reducible to the former; for

$$\frac{-2a^2 + 8ac - 9b^2}{a^2 - 4c} = -2a - \frac{9b^2}{a^2 - 4c};$$

and it is impossible that $a - 4c$ can be negative; for as $-2a^2 + 8ac - 9b^2$ is by hypothesis positive, and a negative, both terms of the second member of this equation would be necessarily positive, while the first member of the equation would be negative.

It follows therefore that when [2] exists [1] must necessarily exist also; and conversely.

Hence, that the roots may all be real, we must have the conditions

$$a < 0, \quad a^2 - 4c > 0, \quad f > 0,$$

and, that they may all be imaginary, we must have, in conjunction with $f > 0$, either

$$a > 0, \quad \text{or} \quad a^2 - 4c < 0.$$

LAGRANGE's equation of the differences is an equation in y , got by eliminating x from the proposed equation

$$f(x) = 0 \dots [A]$$

in conjunction with the equation

$$f'(x) + f''(x) \frac{y}{1 \cdot 2} + f'''(x) \frac{y^2}{1 \cdot 2 \cdot 3} + \dots = 0 \dots [B].$$

(See *Theory of Equations*, p. 214.)

This elimination is effected by seeking the common measure of [A] and [B], the process terminating when we arrive at a remainder in y independent of x : this, it is obvious, will require just as many steps as would bring us to a remainder independent of x when for [A] and [B] we substitute the functions

$$f(x) \text{ and } f'(x)$$

and it is further obvious that this latter remainder would be that which LAGRANGE's remainder would become for $y = 0$. But the remainder spoken of, disregarding sign, is that which constitutes the final function in STURM's theorem. Consequently, with the exception of the sign, LAGRANGE's equation of the differences, which is only his final remainder in y equated to zero, will furnish STURM's final function when the terms in y are expunged; that is, the constant term in the equation of the differences, and which is also the constant in the equation of the squares of the differences, is, abstracting from the sign, the same as the constant or final term in STURM's series of functions.*

LAGRANGE's constant term in the equation of the squares of the differences for the general equation of the fourth degree before quoted, is f ; and we have already

* It is proper to mention that this circumstance is also noticed in a paper entitled "On the Existence of a Relation among the Coefficients of the Equation of the Squares of the Differences of the Roots of an Equation," published in No. VI. of the *Cambridge Mathematical Journal*, a work containing much instructive matter, and one which students of analysis will do well to consult.

seen, without reference to this general truth, that f is the same function of the coefficients as X_0 , when the factor α^2 , introduced at the outset, is suppressed.

In like manner the result of every step of the operation with $[A]$ and $[B]$, that is, the successive remainders will, except sign, become the very same as the corresponding remainders of STURM; provided that in each of the former y be made zero.

(131.) No one has ever attempted to exhibit the conditions for ascertaining the number of the roots beyond the point reached by WARING; for LAGRANGE's admission, before quoted in the text, that for the fifth degree even, the computations requisite for this purpose are excessively laborious, applies as well to the deduction of the general expressions themselves, as to the numerical calculations implied in them. But by STURM's theorem, carrying on the work of it agreeably to the type we have proposed, the deduction of the general expressions for the sixth, seventh, eighth, or even higher degrees, is a process of but a very moderate degree of labour or difficulty. If the equation of the squares of the differences was ever of any practical value, the general functions of STURM must be of incomparably more value.

(132.) It is pretty obvious that the process for obtaining the equation of the squares of the differences may be conducted in a manner similar to that which has been employed to deduce STURM's functions. As an illustration, we shall take the example given at page 215 of the *Theory of Equations*, remembering that here the remainders are not to have their signs changed.

In the example referred to we are required to find the common measure of the functions.

$$X = x^3 + px + q.$$

$$X_1 = 3x^2 + 3yx + y^2 + p.$$

or, rather, the final remainder which the operation for the common measure leaves.

Taking only the coefficients of x and its powers, the work is as follows :

$$\begin{array}{r}
 3-3y-(y^2+p) \\
 \times \\
 1+0+p+q \\
 \hline
 -y \quad 3y^2+y(y^2+p) \\
 1 \quad -(y^2+p) \\
 3 \quad 3p+3q \\
 \hline
 2(y^2+p)+(y^3+py+3q) \\
 2(y^2+p)-(y^3+py+3q) \\
 \times \\
 3+3y+(y^2+p) \\
 \hline
 3(y^3+py-3q) \quad -3(y^3+py-3q)(y^2+py+3q) \\
 4(y^2+p)^2 \quad 4(y^2+p)^3 \\
 \hline
 4(y^2+p)^3-3(y^3+py-3q)(y^2+py+3q).
 \end{array}$$

Hence

$$4(y^2+p)^3-3(y^3+py-3q)(y^2+py+3q)=0$$

is the final equation in y sought.

In a similar manner may the remainder, independent of x , be obtained in the general theory of *Elimination*. (See *Theory of Equations*, Part II. Chap. III.)

NOTE II. (page 189.)

On Equations containing Equal Roots, or Roots nearly Equal.

(133.) IN investigating the theory of numerical equations, by whatever method, the greatest, and almost the only difficulty of consequence, with which the analyst has to contend, is one which is inseparably connected with the existence of equal, or nearly equal, roots. In the ordinary methods this difficulty is increased in consequence of those methods furnishing no ready criterion for distinguishing such roots from those that may be imaginary; since the same indications often present themselves, whether the roots be actually equal, close together but unequal, or imaginary.

The unabridged process of STURM leads to perfectly conclusive information upon all these particulars. And it is plain that even the results of the abbreviated method can never leave us in any doubt as to what roots are real and what imaginary. The precluding of all doubt on this head we cannot but regard as the perfection of any method for analyzing a numerical equation. If the number of the imaginary roots, and the leading figures of all the real roots, are infallibly discovered in all cases, everything which calculation can require, such a method, in conjunction with a sure process for the indefinite development of each root, is competent to supply.

The abbreviated mode of conducting the work of STURM's theorem is clearly infallible in supplying the leading figures of all the real roots, be they equal or unequal; and Mr. HORNER's process of continuous approximation makes known the subsequent figures of each root to as remote

a decimal as we please, when such root is incommensurable.

If from STURM's theorem we learn that two or more roots commence with the same figures, HORNER's further development will either separate them, if they are separable, or will show, up to any extent of decimals we may assign, that the roots continue equal.

To approximate to a root to the extent of seven or eight places of decimals, even in an equation of the seventh or eighth degree, is certainly not a formidable task; and if two or more roots keep together throughout this range, they may, no doubt, be pronounced equal, even supposing that they really separate at a decimal still more remote.

If, however, we desire to bring the matter to an infallible test, we may do so by restoring the latter steps of STURM's theorem to their unabridged form; when the sought information will of course be obtained.

But this method of proceeding, if it were absolutely necessary, would imply an imperfection in the abbreviated process, from which it is in reality free. We advert to it simply because it is the most obvious mode of removing the doubt in question; and because, moreover, as the final function tends to zero, the equation, in the case supposed, contains within itself a check to the advance of the figures in the closing steps, usually of sufficient power to render extraneous restrictions unnecessary; so that the work of these steps might be carried on without abbreviation, and yet without any risk of a continued increase of figures, as in the case of unequal and widely different roots.

We shall show, however, in the course of this note, that our abbreviated results are in all cases fully effective; and that the corrections, here adverted to, are not at all necessary to the completeness of the method.

(134.) As an example of equal roots, we shall give the following, from the work of M. MINDY before quoted:

$$X = x^5 - 7x^4 + 4x^3 + 7x^2 - 2x - 2 = 0$$

$$X_1 = 3x^5 - 14x^3 + 6x^2 + 7x - 1$$

$$\begin{array}{r} 3+0 \quad +14 \quad -6 \quad -7 \quad +1 \\ \times \\ 1+0 \quad -7 \quad +4 \quad +7 \quad -2 \quad -2 \end{array}$$

$$\begin{array}{r} 0 \left| \begin{array}{r} 14 \quad -6 \quad -7 \quad +1 \\ 3-21 \quad +12 \quad +21 \quad -6 \quad -6 \end{array} \right. \\ \hline 7 \quad -6 \quad -14 \quad +5 \quad +6 \end{array}$$

$$\begin{array}{r} 7+6 \quad +14 \quad -5 \quad -6 \end{array}$$

$$\begin{array}{r} \times \\ 3-0 \quad -14 \quad +6 \quad +7 \quad -1 \end{array}$$

$$\begin{array}{r} 18 \left| \begin{array}{r} 108+252 \quad -90-108 \\ 21 \quad 294-105-126 \\ 49 \quad -686+294+343 \quad -49 \end{array} \right. \end{array}$$

$$\begin{array}{r} 284-441-127+157 \end{array}$$

$$\begin{array}{r} 284+441+127-157 \end{array}$$

$$\begin{array}{r} \times \\ 7-6 \quad -14+5+6 \end{array}$$

$$\begin{array}{r} 1383 \left| \begin{array}{r} 609903+175641-217131 \\ 1988 \quad 252476-312116 \\ 284^3 \quad -1129184+403280+483936 \end{array} \right. \end{array}$$

$$\begin{array}{r} 266805-266805-266805 \end{array}$$

or

$$\begin{array}{r} 1 \quad - \quad 1 \quad -1 \end{array}$$

$$\begin{array}{r} 1+1 \quad +1 \end{array}$$

$$\begin{array}{r} \times \\ 284-441-127+157 \end{array}$$

$$\begin{array}{r} -157 \left| \begin{array}{r} -157 \quad -157 \\ 284 \quad 284 \\ 1 \quad -127 \quad +157 \end{array} \right. \end{array}$$

$$\begin{array}{r} 0 \quad 0 \end{array}$$

Consequently,

$$X = x^6 - 7x^4 + 4x^3 + 7x^2 - 2x - 2$$

$$X_1 = 3x^3 - 14x^2 + 6x^2 + 7x - 1$$

$$X_2 = 7x^4 - 6x^3 - 14x^2 + 5x + 6$$

$$X_3 = 284x^3 - 441x^2 - 127x + 157$$

$$X_4 = x^2 - x - 1$$

$$X_5 = 0.$$

The equation proposed has therefore *two pairs* of equal roots. They are contained *singly* in the equation

$$x^2 - x - 1 = 0,$$

and are therefore easily determined. Dividing the function X by $(x^2 - x - 1)^2$, or

$$x^4 - 2x^3 + x^2 + 2x + 1,$$

the quotient is $x^2 + 2x - 2$. Hence the remaining roots of the proposed equation are those of the quadratic

$$x^2 + 2x - 2 = 0.$$

In this example a very decided indication of equal roots, or of a function by which the original polynomial is actually divisible, is furnished at the close of the third step of the work. And as our abbreviations never commence till the numbers in the course of derivation threaten to become inconveniently large or unmanageable, in a case like this these abbreviations would never be introduced;

but the work would be allowed to proceed, unabridged, to the close.

(135.) It will often happen, however, in applying our method to the higher equations, that, on account of the increasing magnitude of the numbers, we shall commence our abbreviations before these indications of equal roots can discover themselves, these latter occurring at a more advanced step, and after we have ceased to attend to the common factors in our several results for the purpose of reducing their magnitude.

We shall give an example of this, and then show, finally, how, by disregarding all these indications, and carrying on our process up to the close of the work, as in the examples already given in the text, we may obtain, free from every ambiguity, a complete analysis of the equation under examination.

The following equation is quoted by CAGNOLI from a Memoir on Equations by RUFFINI, whose work, however, we have not seen.

Required the analysis of the equation

$$x^6 - 16x^5 + 85x^4 - 144x^3 - 57x^2 + 126x + 54 = 0.$$

We shall work the first step of this equation conformably to the shorter method explained at p. 209.

The coefficients in X and X_1 , those of the latter function being divided by 2, are

$$1 - 16 + 85 - 144 - 57 + 126 + 54$$

$$3 - 40 + 170 - 216 - 57 + 63,$$

and the work is as follows :

$$\begin{array}{r}
3 \mid -40 + 170 - 216 - 57 + 63 \\
-16 \mid -170 + 432 + 228 - 630 - 324 \\
\hline
640 - 2720 + 3456 + 912 - 1008 \\
-510 + 1296 + 684 - 1890 - 972 \\
\hline
130 - 1424 + 4140 - 978 - 1980 \\
\text{or } \div 2 \\
65 - 712 + 2070 - 489 - 990 \\
\hline
65 + 712 - 2070 + 489 + 990 \\
\begin{array}{r}
\times \\
3 - 40 + 170 - 216 - 57 + 63
\end{array} \\
\hline
-464 \mid -330368 + 960480 - 226896 - 459360 \\
195 \mid -403650 + 95355 + 193050 \\
65 \mid 718250 - 912600 - 240825 + 266175 \\
\hline
15768 - 143235 + 274671 + 193185 \\
\text{or } \div 3^3 \\
584 - 5305 + 10173 + 7155 \\
\hline
584 + 5305 - 10173 - 7155 \\
\begin{array}{r}
\times \\
65 - 712 + 2070 - 489 - 99
\end{array} \\
\hline
-121 \cdot 5462 \mid -644803 + 1236489 + 869663 \\
65 \mid -661245 - 465075 \\
584 \mid 1208880 - 285576 - 578160 \\
\hline
97168 - 485838 - 291503 \\
\hline
97168 + 485838 + 291503 \\
\begin{array}{r}
\times \\
584 - 5305 + 10173 + 7155
\end{array} \\
\hline
-2385 \mid -11587 - 6952 \\
584 \mid 1702 \\
97168 \mid 9885 + 6952 \\
\hline
0 \quad 0 \\
\hline
\end{array}$$

From this last result we may conclude that the equation has equal roots, these roots being contained *singly* in the preceding function equated to zero. This preceding function must therefore be a divisor of the original polynomial; and consequently the numbers, at present involved in it, must admit of a very large common divisor. In fact, as the leading coefficient of the original polynomial is *unity*, the leading coefficient of the reduced divisor must be unity also: hence the three numbers in the function referred to must be divisible by the first of those three. Actually dividing them by this first, namely, by 97168, we find for our divisor the expression

$$x^2 - 5x - 3.$$

It is true that 5 times our divisor do not rigorously give 485838, but 485840, differing from it by two units; nor do 3 times the same give 291503, but 291504. These minute differences are of course to be expected, on account of the rejected figures to the right. But we conclude, with as much confidence as if these figures were actually preserved, that the function in question, when reduced by dividing by the leading coefficient, can be no other than that just obtained.

There may, however, seem to be a bare possibility that our final result might turn out to be a minute finite quantity, had there been no curtailments; and that therefore, however near to equality certain roots may be, they may nevertheless not be rigorously equal; nor, in consequence, the reduced expression above rigorously a divisor of the original polynomial.

But every scruple of this kind is removed by the final step which completes the analysis of our equation; that is, by the actual division of X by $x^2 - 5x - 3$, or rather by the *square* of this.

The result of this division is .

$$x^3 - 6x + 6,$$

and hence our equation is made up of the three quadratics

$$x^2 - 5x - 3 = 0$$

$$x^2 - 5x - 3 = 0$$

$$x^2 - 6x + 6 = 0$$

and therefore its six roots are

$$\frac{5 \pm \sqrt{37}}{2}, \quad \frac{5 \pm \sqrt{37}}{2}, \quad 3 \pm \sqrt{3}.$$

If our final result had not been accurately zero ; and, in consequence, the roots supposed equal had really a minute difference, the circumstance would have been unequivocally made known by the occurrence of a *remainder* after the division by the supposed common measure, or complete divisor employed above.

(136.) It is this that removes all objection to the abbreviations which so greatly reduce the labour of STURM's theorem. We need never remain in any perplexity at the close of the abbreviated process, as to whether, from the character of our final result, roots absolutely equal are indicated, or roots differing from each other by only a minute quantity. If the numbers in this final result have wholly disappeared, or have dwindled down each to a mere unit, or at most to two or three units, the great probability

is that *equal* roots exist. If such be the case, the immediately preceding function may be so reduced, by the suppression in it of a common factor, that the extreme coefficients in it may become divisors of those in the original polynomial, each of each. If, however, such common factor does not exist, we conclude at once that the roots whose character we seek are not equal, but only nearly so; and this preceding function, just adverted to, will supply us with the leading figures of those roots, each root entering *twice* into the original equation.

The number and situations of all the real roots are then to be determined as follows; the leading figures, that is, the situations of the roots of the function last deduced being known:

Suppose this last function, or that furnishing the first figures of the nearly equal roots to be X_n ; and, as usual, write the series

$$X, X_1, X_2, \dots X_n.$$

Then, since we know between what numbers all the roots of X_n are situated, we know how often X_n vanishes, in substituting the numbers $0, 1, 2, \dots \infty$, and the numbers $0, -1, -2, \dots -\infty$, for x .* Consequently, we always know how many variations are lost (or gained) in the series

$$X_n, X_{n+1}, \dots$$

* It will be remembered that when we know the number of roots which any intermediate function has between a and b , we at the same time know how many variations are lost in the series, from this function to the end, by the substitution of a and b . For these lost variations, by the nature of STURM'S theorem, must equal in number the roots spoken of; so that, knowing the places of these roots, we shall have no occasion for the subsequent functions themselves.

up to the end, without requiring the actual exhibition of this series. This follows from the very nature of STURM's theorem. Hence the number and situations of the real roots are easily determinable without the aid of the functions beyond X_m .

As to the situations of the roots of $X_m = 0$, supposed to be previously ascertained, they may always be discovered with but little trouble. If $X_m = 0$ be a simple equation, a mere inspection of it is sufficient; if it be a quadratic, the increase of labour is very trifling; and the same may be said in the case of $X_m = 0$ being a cubic equation, provided we adopt in the enquiry the short method explained at p. 209. It would be useless to advert to the case of $X_m = 0$ being an equation of the fourth degree; or to occupy ourselves about supposed equations having eight roots differing from one another only at an advanced figure.

Let us suppose, now, that the coefficients in X_m really have a common factor, and one of sufficient magnitude to reduce the leading and final numbers in X_m to the smallness of factors of the corresponding numbers in X . Divide X by the square of X_m thus reduced. If the remainder is zero, $X_m = 0$ furnishes *singly* the equal roots of the proposed equation. If the remainder is not zero, the same equation will, as before, involve *singly* the roots nearly equal; and we shall conclude that none absolutely equal has place in the equation.*

* We all along presume that the common divisor, when such really exists, has not itself equal roots. When this divisor is of no higher degree than the second, an inspection of the coefficients in it is sufficient to make known whether its simple factors are equal or not. But, when the divisor is of a higher degree than the second, its composition cannot be so readily discovered. In this case, therefore, we should divide X simply by X_m , and not by its square, since X is divisible by the square of X_m only when the roots of $X_m = 0$ are unequal. The greatest common divisor of X and X_1 is always made up of the

(137.) We have now pointed out the method of proceeding, in order to remove all ambiguity in a case, the actual occurrence of which in practice we regard as in the highest degree improbable; the occurrence, namely, of roots so exceedingly close together, as to render it doubtful whether our final function, preserved true to four or five places of figures, indicates, by the actual evanescence of those figures, equal roots or not.

Let it be remembered that the function preceding this, true to a still greater number of figures, must in case of *unequal* roots imply the coincidence of these roots to the extent of at least five or six places of figures. We can give no examples of equations of this kind to illustrate the precepts just laid down, for we have never met with an equation containing roots which keep thus far together and then separate at a more advanced figure. Nor do we know in what manner to frame such an example, at least one of a moderately high degree, except by the introduction of very large coefficients into the original equation. Nevertheless, the method here explained provides satisfactorily even for extreme cases of this kind; so that the abbreviated mode of deducing STURM's functions, while it renders his theorem practicable in its application to the higher equations, pre-

multiple factors of X , each raised to a power a unit lower than that with which the same factor enters X .

If in the case of roots very nearly equal, that is, in the case where all the coefficients in the function next to X_m become nearly evanescent, it should so happen that there are also roots in $X_m = 0$ so nearly equal as to render it doubtful whether a pair of them may be real or imaginary, we should proceed as if they were real, and determine, as explained in the text, the variations lost, in the interval which comprises them, in both the partial series from X to X_m , and from X_m to the end. If our supposition be correct, it will be verified by these variations. When X_m is a quadratic, there should be a loss of four or at least of three variations in the assumed interval, and the same when X_m is a cubic, otherwise the roots tested are imaginary.

serves, at the same time, all the accuracy and certainty that belong to the unabridged process.

(138.) It seems, then, that there remains but one more point to be established; and which is, that in the case of equal roots the function preceding the last, when the common factors are expunged, does really present coefficients remarkably small; the first and last never exceeding the first and last in the proposed equation, but always being factors of them, each of each.

To prove this let the original polynomial be

$$ax^n + bx^{n-1} + cx^{n-2} + \dots k \dots [1],$$

and let the function in STURM's series that divides this without remainder be

$$a'x^m + b'x^{m-1} + \dots k' \dots [2].$$

The coefficients in [1] have been deprived of all common factors, so have those in [2]. This being borne in mind, let it be supposed that in order to perform the division of [1] by [2], and at the same time to preserve the quotient free from fractions, it be necessary to multiply [1] by some factor P. Let the quotient of the division, freed from fractions by this preparatory operation, be

$$a''x^p + b''x^{p-1} + \dots k'' \dots [3];$$

this quotient has of course no common factor; otherwise the multiplier introduced into [1] would contain that factor and

would therefore be needlessly large. Now, as the operation terminates without remainder, we have

$$[2] \times [3] = P \times [1].$$

The second member of this equation is divisible by the number P : hence the first member must be divisible by P .* But neither $[2]$ nor $[3]$ are divisible by any number but unity. Consequently $P = 1$.

It follows therefore that $[1]$ is accurately divisible by $[2]$ without any preparation, giving a quotient free from fractions. Hence a' is either equal to, or else a factor of a ; and k' either equal to or a factor of k .

As in the case of equal roots, entering in *pairs*, the original polynomial is divisible, not only by the last of STURM's functions, but also by the square of it, common factors being suppressed, we see that the extreme numbers in the original polynomial must be divisible by the squares of the extreme numbers in the final function, each of each, if such equal roots really exist. And, more than this, the extreme numbers in every one of the preceding functions must be divisible by the corresponding extremes of the common divisor, although not necessarily by their squares.

It is scarcely necessary to add that when the division of $[1]$ by $[2]$ leaves a remainder, the more minute this remainder is, the nearer does $[2]$ approach to a measure of $[1]$.

(139.) We shall merely make one or two observations in conclusion, chiefly for the purpose of showing the universality of a principle of useful application in the process

* If any rational and integral polynomial be divisible by a number P , then one, at least, of the integral factors whose product give that polynomial must be divisible by the same number P . (See BOURDON, *Algèbre*, p. 614, edit. 1837.)

for the common measure, but which has hitherto been restricted to particular cases.

It is a condition stipulated for by STURM, in the investigation of his theorem, that equal roots be first removed from the equation. He afterwards shows, however, that the presence of such roots will not neutralize his method, although he gives no examples in which they enter. Nearly all who have discussed and exemplified the theorem, expressly exclude equal roots from their examples. Now we would remark here that, with this restriction, much of the labour which these authors bestow on their examples might be spared. In all cases the derivation might terminate when the quadratic function is reached; for it will be easy to discover, by a glance at the coefficients, if the roots of this quadratic are equal, and if so, to find the common leading figure; or if the roots are unequal, the places of each may be easily found by a few trials; after which all the necessary particulars may be determined without the aid of the subsequent functions, as explained above.

(140.) STURM himself has not been inattentive to this circumstance. For he observes, "Whenever in seeking the greatest common divisor of V and V_1 , we shall arrive at a polynomial V_n (one of the second degree for example,) which equated to zero, will give only imaginary values of x , it will not be necessary to carry on the divisions further, for this polynomial V_n will constantly be of the same sign as its first term for all the real values of x , so that it may be taken for the last of the auxiliary functions V_1, V_2 , &c. We might, indeed, even stop at a polynomial V_n , which would be destroyed for real values of x , provided that we could determine all these values. For representing by $p, q, r, \dots$ those of such values which would be comprised between A and B , after having arranged them in the order of their magnitude, commencing with the least, and observing

that V_n preserves the same sign for all values of x comprised between A and p , we should find, by the application of the theorem, how many roots the equation $V = 0$ has between A and $p-u$, u being a very small quantity; in like manner, V_n having also a constant sign for all values of x comprised between p and q , we should find how many roots $V=0$ has between $p+u$ and $q-u$, that is, between p and q , by taking u sufficiently small: we should in a similar manner ascertain how many roots $V=0$ has between q and r , and so on. This is on the supposition that the equation $V=0$ has no equal roots, or that a value of x , which destroys V_n , does not at the same time destroy V .

These cases, in which we may diminish the number of auxiliary functions, deserve particular attention, for the calculations necessary for the determination of the functions $V_n, V_{n-1}, \dots$ are very long, especially when we arrive at the last functions, in consequence of the magnitude of the numerical coefficients.*

(141.) To this we may add that, without any hypothesis as to equal roots entering the equation, we may in all cases proceed upon the above principle, when we have reached a quadratic function having imaginary roots; and we have accordingly resorted to it in some of the earlier examples in this dissertation, without any previous stipulation as to the nature of the roots. STURM does not appear to have perceived the universality of this principle; but it may be established as follows:

Suppose the quadratic function itself is a divisor of the proposed polynomial X , then this function, being the only common divisor of X and X_1 , the division of X by it will be a polynomial X_n , containing the remaining roots, which roots will be all unequal.

Now it is plain, from the nature of the process, that the

* SPILLER's translation of STURM's *Mémoire*, p. 15.

several functions or remainders derived from X_n , as a primitive, will be respectively equal to the corresponding remainders derived from X , divided each by the quadratic function common to them all; which function is, by hypothesis, always *positive*. Hence there must be the same variations of sign in the functions arising from X as in those arising from X_n . Consequently the former series of functions will determine the number and places of the real roots of $X=0$, whether the final quadratic, with imaginary roots, divides X or not.*

That the function of the first degree, which would be given by continuing the operation a step further, cannot divide X is obvious; because, if so, it would also divide the quadratic itself, which would thus have a real root. When therefore the operation terminates in a quadratic with imaginary roots, we may conclude that the proposed equation cannot have equal roots unless they be imaginary also; and that the determination of the real roots is altogether independent of this circumstance.

(142.) This is a useful truth, not only as respects the method of STURM, but in reference to the labour attending the practice of the common measure in general. It is always easy to see whether or not the extreme terms of any quadratic function give a product which exceeds the fourth part of the square of the mean. If we find that they do give such a product, then we shall know that it is useless to extend the work; for if any common measure exist, it must be the quadratic expression at which we have arrived; and in order that this may be a common measure, the first term in it must be a divisor of the first term of every one of the remainders preceding it, as well as of the

* The same reasoning will evidently apply whatever be the degree of the function containing only imaginary roots; but the property is of little use except in the case of functions of the second degree, on account of the difficulty of applying the necessary tests to the expression.

first terms of the original polynomials; the last term must in like manner divide the last terms of the same expressions. The actual division however, by the supposed common divisor, is generally attended with but very little trouble; far less than would be requisite to prolong the operation for the common measure one step further.

(143.) We might extend the principle just established to the case in which the quadratic arrived at in STURM'S method had real roots, without making any preliminary condition as to equal roots. For whether or not this quadratic function is itself a divisor of the proposed polynomial may be ascertained as already directed; and, on the supposition that the roots of the proposed are unequal, we may discover their number and situations by help of the places which the roots of this quadratic occupy, as explained at page 233.

But if this supposition should prove wrong, the equation really having a pair of equal roots, which would have been discovered by continuing the work beyond the quadratic, then we shall find one real root unaccounted for by our previous examination; so that we shall immediately infer the existence of an additional root between one of the pairs of limits which that examination makes known. The proper pair is found by substituting each pair in the original polynomial; for if the place of the wanting root is between limits already comprehending an odd number of roots, the substitution of these limits will give like signs; but if its place be among an even number of roots, the limits will give unlike signs. Its proper situation may therefore be easily discovered.

(144.) In equations of a very high degree, those of the ninth or tenth degree for instance, it will generally be found convenient to make use of this principle; postponing our abbreviations to no later a period in the work than may

be necessary to provide against ambiguity in the function of the second degree ; to close the operation at that step, and then to proceed as just described.

If, for example, we had adopted this course in the equation of the seventh degree, examined at page 195, and had made the fourth step of the work the last, we might have abridged that step more freely ; computing it thus :

$$\begin{array}{r}
 -1018178 + 664097 \quad + \quad 8723 - 37487 \\
 \quad \quad \quad \times \\
 \quad \quad \quad 28665 + 3098760 + 2009931 + 23618 - 113876 \\
 \hline
 -3080074 \left| \begin{array}{l} -204547 - 2687 + 11546 \\ -28665 \quad -25 \quad +107 \\ 1018178 \quad 204647 + 2405 - 11595 \end{array} \right. \\
 \hline
 \quad \quad \quad -75 \quad +175 \quad + \quad 49 \\
 \hline
 \end{array}$$

The quadratic function in this case is therefore

$$X_3 = -75x^2 + 175x + 49.$$

Equating this to zero, the roots are easily found to lie, one between 0 and -1 , and one between 2 and 3. Hence the functions

$$X_2, X_6, X_7,$$

gain two variations from $x = -\infty$ to $x = +\infty$, seeing that the leading sign of X_3 is minus.

Also the functions

$$X \quad X_1 \quad X_2 \quad X_3 \quad X_4 \quad X_5$$

giving for $x = -\infty$, $- \quad + \quad - \quad + \quad + \quad -$

and for $x = +\infty$, $+ \quad + \quad + \quad + \quad - \quad -$

lose three variations within the same limits. Consequently, upon the whole, only *one* variation is lost between the limits $-\infty$ and $+\infty$, in the entire series of functions.

$$X, X_1, \dots, X_n,$$

and therefore the equation has but one real root. The place of this is found, from the original equation, to lie between 2 and 3, as before.

The actual determination of the situations of the two roots of $X_1 = 0$ happens to be, in this case, superfluous; as the equation turns out to have but a single real root, the place of which is ascertained without any aid from the derived functions. But when several real roots are found to exist, then the actual situations of them can be discovered only when we know the particular intervals between which the changes of sign in our final function happen; that is to say, only when we know whereabouts the roots of this final function are situated.

All this however has been sufficiently explained already.

(145.) We remark, finally, that the method of proceeding just explained will, in most cases, greatly shorten the work, even when no numerical abbreviations are introduced.

We have already seen, page 214, that a biquadratic equation may be completely analyzed by employing the general method of derivation in one step only. From what has now been said, we see that even this step may be dispensed with, without any danger of committing error from the presence of equal roots, whether these be real or imaginary. This is a circumstance well worthy of attention; and it was with especial reference to the facilities thus introduced into the analysis of equations of the fourth degree, that we characterized the operation for separating the roots of a biquadratic equation, at page 215, as one of so little labour or difficulty.

In fact the observations in article (126), setting forth the extreme simplicity to which the problem of analysing a cubic equation is now reduced, may, with equal propriety, be extended to equations of the fourth degree. For, as in cubics so here, the single function X_3 , in conjunction with the two preceding ones, immediately given by the proposed equation, is all that is indispensably necessary for the separation of the roots; and we have already seen with what ease the function X_4 may always be computed. On this account the analysis and solution of equations of the third and fourth degrees might with propriety be detached from the general theory, and incorporated among the elementary operations of common algebra.

Instead therefore of the THEOREM OF STURM being impracticable beyond equations of the fourth degree, as some authors have stated it to be, on account of the labour attending the numerical computations, we may truly affirm that, under the form in which we have here presented it, it is not till we have passed this limit that any computation, worth mentioning, is really indispensable to the complete analysis of the equation.

Even in equations of the fifth degree, the function of the second order, at which we see the process of derivation may be made to terminate, may generally be reached without any abbreviations being employed; and yet with but a very moderate amount of numerical computation.

In equations of the sixth and of a more advanced degree, it will usually be found expedient to resort to the abbreviated method before the quadratic function is arrived at. The only object of this abbreviated method being to restrain the numerical results within manageable bounds, it should of course always be delayed till these results threaten to become inconveniently large; and when it is introduced it must be employed with that discretion which it is requisite to exercise in all abbreviated methods; that is to say, our curtailments must be made with such

moderation that our abridged forms may still retain all the important peculiarities of the complete functions themselves. The precepts and examples already given will, we think, furnish guidance sufficient, in this matter for every case that can occur in actual practice.

Having thus settled the point at which the abbreviations are to commence, we may then either carry on the several steps of the work till it terminates of itself, as in most of the examples given in the text; or else, conformably to the directions in the present note, we may discontinue the process of derivation when the function of the second degree is obtained, and then complete the analysis of the equation in the manner which has just been explained.

SUPPLEMENTARY NOTE ON THE BINOMIAL
THEOREM.

(146.) IN the third edition of the *Elementary Treatise on Algebra*, recently published, the author has offered an investigation of the Binomial Theorem, which he conceives to have some claims to originality. That part of the proof, however, which establishes the law of the coefficients, and which commences at page 183, has appeared before; at least so much of it as follows equation [8]. This he has virtually acknowledged at page x of the preface to the work itself. Very soon after the publication of that treatise, the author perceived that the part of the proof which he had borrowed might be superseded by a much more simple and obvious method of deduction; and,

as he has here a blank page at his disposal, he feels induced to occupy it with the improvement which then suggested itself.

From equation [8], page 183, we have

$$m(1+b)^{n-1} = m + 2Qb + 3Rb^2 + 4Sb^3 + \&c.$$

Also from [6], page 182,

$$(1+b)^n = 1 + mb + Qb^2 + Rb^3 + Sb^4 + \&c.$$

Hence, multiplying this by m and the former by $(1+b)$, and then equating the right-hand member of each, we have

$$\begin{aligned} m + m^2b + mQb^2 + mRb^3 + mSb^4 + \&c. \\ = m + (2Q+m)b + (3R+2Q)b^2 + (4S+3R)b^3 + (5T+4S)b^4 + \&c. \end{aligned}$$

Consequently

$$\begin{aligned} 2Q + m &= m^2 \quad \therefore Q = \frac{m(m-1)}{2} \\ 3R + 2Q &= mQ \quad \therefore R = \frac{m(m-1)(m-2)}{2 \cdot 3} \\ 4S + 3R &= mR \quad \therefore S = \frac{m(m-1)(m-2)(m-3)}{2 \cdot 3 \cdot 4}; \end{aligned}$$

&c. &c. &c.

which establishes the law of the coefficients Q, R, S , &c.



ANALYTICAL COURSE OF MATHEMATICS,

BY J. R. YOUNG,

PROFESSOR OF MATHEMATICS, ROYAL COLLEGE, BELFAST.

I.

ELEMENTS OF GEOMETRY; containing a New and Universal Treatise on the Doctrine of Proportion, together with Notes, in which are pointed out and corrected several important Errors that have hitherto remained unnoticed in the Writings of Geometers. 8vo, 8s.

II.

AN ELEMENTARY TREATISE ON ALGEBRA, Theoretical and Practical; with attempts to simplify some of the more difficult parts of the Science, particularly the Demonstration of the Binomial Theorem in its most general form; the Summation of Infinite Series, &c. The third Edition, 6s.

III.

A KEY to the above. New Edit. by W. H. SPILLER. 6s.

IV.

THE GENERAL THEORY AND SOLUTION OF ALGEBRAICAL EQUATIONS, in which are embodied the recent Researches of Budan, Fourier, and Horner; together with a complete Solution of the important Problem of the Separation of the Real and Imaginary Roots of an Equation, under all circumstances; being the substance of the remarkable paper by M. STURM, printed among the *Mémoires présentés par divers Savans à l'Académie Royale des Sciences de l'Institut de France*. 12mo, 9s.

V.

ELEMENTS OF PLANE AND SPHERICAL TRIGONOMETRY, with its Applications to the Principles of Navigation and Nautical Astronomy, with the necessary Logarithmic and Trigonometrical Tables. By J. R. YOUNG. (To which is added, some Original Researches in Spherical Geometry; by T. S. DAVIES, F.R.S.E. F.R.S.A.) 6s. cloth.

VI.

MATHEMATICAL TABLES; comprehending the Logarithms of all Numbers from 1 to 30,600; also the Natural and Logarithmic Sines and Tangents; computed to seven places of Decimals, and arranged on an improved Plan; with several other Tables, useful in Navigation and Nautical Astronomy, and in other departments of Practical Mathematics. 6s. 6d. cloth.

VII.

AN ELEMENTARY TREATISE ON THE COMPUTATION OF LOGARITHMS. Intended as a Supplement to the various Books on Algebra. A new edition, enlarged. 12mo, 5s.

VIII.

ANALYTICAL GEOMETRY, Part I. on the Theory of the Conic Sections. Second Edition. 12mo, 6s. 6d.

IX.

ANALYTICAL GEOMETRY, Part II. on the General Theory of Curves and Surfaces of the Second Order. Second Edition. 12mo, 7s. 6d.

Souter's List of Mathematical Works.

X.

ELEMENTS OF THE DIFFERENTIAL CALCULUS : comprehending the General Theory of Curve Surfaces and of Curves of Double Curvature. Second Edition. 12mo, 9s. in cloth.

Also an Octavo Edition for the Universities. 12s.

XI.

ELEMENTS OF THE INTEGRAL CALCULUS : with its Applications to Geometry, and to the Summation of Infinite Series, &c. 9s. in cloth.

XII.

ELEMENTS OF MECHANICS ; comprehending the Theory of Equilibrium and of Motion, and the first Principles of Physical Astronomy, together with a variety of Statical and Dynamical Problems. Illustrated by numerous Engravings. 10s. 6d. in cloth.

XIII.

A LECTURE ON THE STUDY OF MATHEMATICS. 2s. 6d.

XIV.

A CATECHISM OF ALGEBRA ; Part I. Being an easy Introduction to the first Principles of Analytical Calculation. 9d. sewed, or 1s. bound.

XV.

A CATECHISM OF ALGEBRA ; Part II. 9d. sewed, or 1s. bound.

XVI.

EUCLID'S ELEMENTS : the first Six and the Eleventh and Twelfth Books. Chiefly from the Texts of SIMSON and PLAYFAIR. With Corrections, and an Improved Fifth Book ; also Supplementary Treatises on Plane Trigonometry, Incommensurable Quantities, the Composition of Ratios, and the Quadrature of the Circle : together with Notes and Comments. Designed for the use of Colleges and Private Students. Second Edition, considerably enlarged. 5s. cloth.

Also lately published,

LACROIX'S ELEMENTS OF ALGEBRA ; translated from the French. By W. H. SPILLER. 12mo, 7s. 6d.

ON THE SOLUTION OF NUMERICAL EQUATIONS. By C. STURM. Translated from the *Mémoires présentés par divers Savans à l'Académie Royale des Sciences de l'Institut de France*, by W. H. SPILLER. 4to, sewed, 7s. 6d.

AN ESSAY ON MUSICAL INTERVALS, HARMONICS, and TEMPERAMENT ; in which the most delicate, interesting, and most useful parts of the Theory are explained, in a manner adapted to the comprehension of the practical Musician. By W. S. B. WOOLHOUSE. 12mo, 5s.

A BRIEF TREATISE ON THE USE AND CONSTRUCTION of a CASE of MATHEMATICAL INSTRUMENTS. By GEORGE PHILLIPS, B.A. Queen's Coll. Cambridge. New edition. 2s. 6d.

JOHN SOUTER, 131, FLEET STREET, LONDON.



NAPOLI

261

I

B. Prov.

NAZIONALE

VITT. EM. III



